CONTENTS

PREFACE

MAJOR CHANGES IN THE THIRD EDITION

We have made the following major changes in this third edition of *Managerial Decision Modeling with Spreadsheets*:

- *Updated from Excel 2003 to Excel 2007/2010*—All spreadsheet applications have been fully updated to Excel 2010. Likewise, the illustration of Microsoft Project and Crystal Ball has been updated to their latest versions. The software program ExcelModules that accompanies this book has also been updated to suit Excel 2010 as well as 32-bit and 64-bit systems.
- *Significant number of revised and new end-of-chapter exercises*—We have made revisions to many of the end-of-chapter exercises from the previous edition and have added several new exercises. On average, there are more than 45 end-of-chapter exercises per chapter. Many of these exercises include multipart questions, giving instructors a rich pool of questions from which to select.
- *More challenging chapter examples and end-of-chapter exercises*—Many of the chapter examples and end-of-chapter exercises have been revised to make them more current, rigorous, and better suited to a computer-based solution environment.
- *Expanded use of color in Excel screenshots*—Unlike the first two editions where only the first six chapters had screenshots in color, screenshots in all chapters are now in color. This allows greater clarity in our explanations of these screenshots and their understanding by students.
- *Updated* DM in Action *Boxes*—Now includes 27 new *DM in Action* boxes that illustrate the use of decision modeling in real-world scenarios, most of them in well-known global organizations. Many of these examples are from recent issues of *Interfaces* and discuss applications that have occurred within the past few years.
- *Expanded coverage of unbalanced transportation and transshipment models (Chapter 5)*—This edition includes discussion of a larger transshipment example with pure transshipment nodes. We have also discussed how dummy nodes can be used to convert unbalanced models into equivalent balanced models for ease of use.
- *Coverage of OptQuest in simulation (Chapter 10)*—The textbook now discusses the use of the OptQuest procedure available in Crystal Ball to automatically identify the best combination of values for decision variables that optimizes a desired output measure in a simulation model.
- *Expanded discussion of nonlinear programming (Chapter 6)*—The textbook now illustrates how the Multistart option available in Excel 2010 when using the GRG procedure to solve nonlinear programming models can be used to increase the likelihood of identifying a global optimal solution for these models.

OVERVIEW

In recent years, the use of spreadsheets to teach decision modeling (alternatively referred to as *management science*, *operations research*, and *quantitative analysis*) has become standard practice in many business programs. This emphasis has revived interest in the field significantly, and several textbooks have attempted to discuss spreadsheet-based decision modeling.

However, some of these textbooks have become too spreadsheet oriented, focusing more on the spreadsheet commands to use than on the underlying decision model. Other textbooks have maintained their algorithmic approach to decision modeling, adding spreadsheet instructions almost as an afterthought. In the third edition of *Managerial Decision Modeling with Spreadsheets*, we have continued to build on our success with the first two editions in trying to achieve the perfect balance between the decision modeling process and the use of spreadsheets to set up and solve decision models.

It is important that textbooks that support decision modeling courses try to combine the student's power to logically model and analyze diverse decision-making scenarios with software-based solution procedures. Therefore, this third edition continues to focus on teaching the reader the skills needed to apply decision models to different kinds of organizational decision-making situations. The discussions are very application oriented and software based, with a view toward how a manager can effectively apply the models learned here to improve the decision-making process. The primary target audiences for this textbook are students in undergraduate- and graduate-level introductory decision modeling courses in business schools. However, this textbook will also be useful to students in other introductory courses that cover some of the core decision modeling topics, such as linear programming, network modeling, project management, decision analysis, and simulation.

Although the emphasis in this third edition continues to be on using spreadsheets for decision modeling, the textbook remains, at heart, a *decision modeling* textbook. That is, while we use spreadsheets as a tool to quickly set up and solve decision models, our aim is not to teach students how to blindly use a spreadsheet without understanding how and why it works. To accomplish this, we discuss the fundamental concepts, assumptions, and limitations behind each decision modeling technique, show how each decision model works, and illustrate the real-world usefulness of each technique with many applications from both for-profit and not-for-profit organizations.

We have kept the notation, terminology, and equations standard with other textbooks, and we have tried to write a textbook that is easy to understand and use. Basic knowledge of algebra and Excel are the only prerequisites. For your convenience, we have included a brief introduction to Excel 2010 as an appendix.

This textbook's chapters, supplements, and software packages cover virtually every major topic in the decision modeling field and are arranged to provide a distinction between techniques that deal with deterministic environments and those that deal with probabilistic environments. Even though we have produced a somewhat smaller textbook that covers only the most important topics, we have still included more material than most instructors can cover in a typical first course. We hope that the resulting flexibility of topic selection is appreciated by instructors who need to tailor their courses to different audiences and curricula.

OVERALL APPROACH

While writing this third edition, we have continued to adhere to certain themes that have worked very well in the first two editions:

- First, we have tried to separate the discussion of each decision modeling technique into three distinct issues:

  1. Formulation or problem setup
  2. Model solution
  3. Interpretation of the results and what-if analysis

  In this three-step framework, steps 1 and 3 (formulation and interpretation) call upon the manager's expertise. Mastering these steps now will give students a competitive advantage later, in the marketplace, when it is necessary to make business decisions. We therefore emphasize these steps.

- Second, we recognize that business students are primarily going to be users of these decision modeling techniques rather than their developers. Hence, to deal with step 2 (model solution), we have integrated our discussions with software packages so that students can take full advantage of their availability. In this regard, the textbook exploits the wide availability and acceptability of spreadsheet-based software for decision modeling techniques.

Excel is a very important part of what most instructors consider the two main topics in any *basic* decision modeling textbook: linear programming and simulation. However, we recognize that some topics are not well suited for spreadsheet-based software. A case in point is project management, where Excel is generally not the best choice. In such cases, rather than try to force the topic to suit Excel, we have discussed the use of more practical packages, such as Microsoft Project.

- Third, although we use software packages as the primary vehicle to deal with step 2, we try to ensure that students focus on *what* they are doing and *why* they are doing it, rather than just mechanically learning which Excel formula to use or which Excel button to press. To facilitate this, and to avoid the "black box syndrome," we also *briefly* discuss the steps and rationale of the solution process in many cases.
- Fourth, we recognize that in this introductory textbook, the material does not need to be (and should not be) too comprehensive. Our aim here is to inform students about what is available with regard to decision modeling and pique their interest in the subject material. More detailed instruction can follow, if the student chooses, in advanced elective courses that may use more sophisticated software packages.
- Finally, we note that most of the students in decision modeling courses are likely to specialize in *other* functional areas, such as finance, marketing, accounting, operations, and human resources. We therefore try to integrate decision modeling techniques with problems drawn from these different areas so that students can recognize the importance of what they are learning and the potential benefits of using decision modeling in real-world settings. In addition, we have included summaries of selected articles from journals such as *Interfaces* that discuss the actual application of decision modeling techniques to real-world problems.

FEATURES IN THIS TEXTBOOK

The features of the first two editions of this textbook that have been well received as effective aids to the learning process have been updated and expanded in this third edition. We hope that these features will continue to help us to better adhere to the themes listed previously and help students better understand the material. These include the following features:

- *Consistent layout and format for creating effective Excel models*—We use a consistent layout and format for creating spreadsheet models for all linear, integer, goal, and nonlinear programming problems. We strongly believe such a consistent approach is best suited to the beginning student of these types of decision models.
- *Functional use of color in the spreadsheets to clarify and illustrate good spreadsheet modeling*—As part of the consistent layout and format for the spreadsheet models, we have standardized the use of colors so that the various components of the models are easily identifiable. For an excellent illustration of this feature, please see the front end papers of this edition.
- *Description of the algebraic formulation and its spreadsheet implementation for all examples*—For each model, we first discuss the algebraic formulation so that the student can understand the logic and rationale behind the decision model. The spreadsheet implementation then closely follows the algebraic formulation for ease of understanding.
- *Numerous screen captures of Excel outputs, with detailed callouts explaining the important entries*—We have included numerous screen captures of Excel files. Each screenshot has been annotated with detailed callouts explaining the important entries and components of the model. The front end papers of this edition provide a detailed illustration of this feature. Excel files are located at www.pearsonhighered.com/balakrishnan and, for your convenience, the callouts are shown as comments on appropriate cells in these Excel files.
- *Ability to teach topics both with and without the use of additional add-ins or software*—We have discussed several topics so that they can be studied either using Excel's standard built-in commands or using additional Excel add-ins or other software.

For example, we have discussed how Excel's built-in Data Table and Scenario Manager procedures can be used to analyze and replicate even large simulation models. We have also discussed how Crystal Ball can be used to develop models in a more convenient manner, for students who want to install and use this software. Likewise, we have discussed how Microsoft Project can be used to effectively manage large projects.

- *Extensive discussion of linear programming sensitivity analysis, using the Solver report*— The discussion of linear programming sensitivity analysis in this textbook is more comprehensive than that in any competing textbook.
- *Decision Modeling in Action boxes*—These boxes summarize published articles that illustrate how real-world organizations have used decision models to solve problems. This edition includes 27 new *DM in Action* boxes, mostly from recent issues of *Interfaces* that discuss applications that have occurred within the past few years.
- *History boxes*—These boxes briefly describe how some decision modeling techniques were developed.
- *Margin notes*—These notes make it easier for students to understand and remember important points.
- *Glossaries*—A glossary at the end of each chapter defines important terms.
- *ExcelModules*—This program from Professor Howard Weiss of Temple University solves problems and examples in the queuing models (Chapter 9), forecasting models (Chapter 11), and inventory control models (Chapter 12) chapters in this textbook. Students can see the power of this software package in modeling and solving problems in these chapters. ExcelModules is menu driven and easy to use, and it is available at www.pearsonhighered.com/balakrishnan.
- *Microsoft Project*—This software is featured in the project management (Chapter 7) chapter to set up and manage projects. Readers can go to www.microsoft.com to get more information about this popular software.

COMPANION WEBSITE

The following items can be downloaded at www.pearsonhighered.com/balakrishnan:

1. **Data Files—**Excel files for all examples discussed in the textbook (For easy reference, the relevant file names are printed in the margins at appropriate places in the textbook.)
2. **Online Chapter—**The electronic-only Chapter 12: Inventory Control Models in PDF format
3. **Software—**The following software can be downloaded directly from the Companion Website or will link you to the company's website where you can download the free trial version.

   ExcelModules—This program from Professor Howard Weiss of Temple University solves problems and examples in the queuing models (Chapter 9), forecasting models (Chapter 11), and inventory control models (Chapter 12) chapters in this textbook. Students can see the power of this software package in modeling and solving problems in these chapters. ExcelModules is menu driven and easy to use. This can be downloaded directly from the Companion Website.

   TreePlan—This program helps you build a decision tree diagram in an Excel worksheet using dialog boxes. Decision trees are useful for analyzing sequential decision problems under uncertainty. TreePlan automatically includes formulas for summing cash flows to obtain outcome values and for calculating rollback values for determining optimal strategy. This can be downloaded directly from the Companion Website.

   Crystal Ball—This program is a spreadsheet-based application suite for predictive modeling, forecasting, simulation, and optimization. There is a link to the company's website where you can download a free 30-day trial version.

   Risk Solver Platform— This program is a tool for risk analysis, simulation, and optimization in Excel. There is a link to the company's website where you can download a free 15-day trial version.

Subscription Content—a Companion Website Access Code is located on the inside back cover of this book. This code gives you access to the following software:

- *Crystal Ball*— Free 140-day Trial of Crystal Ball Software Compliments of the Crystal Ball Education Alliance.
- *Risk Solver Platform for Education (RSPE)*—This is a special version of Frontline Systems' Risk Solver Platform software for Microsoft Excel.

After redeeming the access code on the back cover of this book you will find a link to each company's website, where you can download the upgrade version.

To redeem the subscription content:

- Visit www.pearsonhighered.com/balakrishnan
- Click on the Companion Website link
- Click on the Subscription Content link
- First-time users will need to register, while returning users may log-in. Enter your access code found on the back cover of this book.
- Once you are logged in you will be brought to a page that will instruct you on how to download the software from the corresponding software company's website.

4. **Key equations—**Files list all the mathematical equations in a chapter.
5. **Internet Case Studies—**There are several additional case studies.

INSTRUCTOR'S SUPPLEMENTS

The supplements to this textbook reflect the spreadsheet emphasis of the textbook and provide students and instructors with the best teaching and resource package available.

- *Companion Website* at www.pearsonhighered.com/balakrishnan—See the detailed list of the contents for the Companion Website above. Instructors can also access solutions to cases on the Website.
- *Instructor's Solutions Manual*—The Instructor's Solutions Manual, prepared by Raju Balakrishnan, includes solutions (along with relevant Excel files) for all end-of-chapter exercises and cases. Also available are the solutions to the Internet Study Cases. The Instructor's Solutions Manual is available for download by visiting the Companion Website.
- *PowerPoint presentations*—An extensive set of PowerPoint slides is available for download by visiting www.pearsonhighered.com/balakrishnan. The slides are oriented toward the learning objectives listed for each chapter and build on key concepts in the textbook.
- *Test Item File*—The Test Item File is available for download by visiting the Companion Website.
- *TestGen*— Pearson Education's test-generating software is available from www.pearsonhighered.com/irc. The software is PC/MAC compatible and preloaded with all the Test Item File questions. You can manually or randomly view questions and drag and drop to create a test. You can add or modify questions as needed.
- *Subscription Content*—A Companion Website Access Code is located on the inside back cover of this book. This code gives you access to the following software:
    - *Crystal Ball*— Free 140-day Trial of Crystal Ball Software Compliments of the Crystal Ball Education Alliance
    - *Risk Solver Platform for Education (RSPE)*—This is a special version of Frontline Systems' Risk Solver Platform software for Microsoft Excel. For further information on **Risk Solver Platform for Education,** contact Frontline Systems at (888) 831-0333 (U.S. and Canada), 775-831-0300, or academic@solver.com. They will be pleased to provide **free evaluation licenses** to faculty members considering adoption of the software. They can help you with conversion of simulation models you might have created with other software to work with Risk Solver Platform (it's very straightforward).

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It is no secret that unlike courses in functional areas such as finance, marketing, and accounting, decision modeling courses always face an uphill battle in getting students interested and excited about the material. We hope that this textbook will be an ally to all in this endeavor.

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CHAPTER 1

Introduction to Managerial Decision Modeling

LEARNING OBJECTIVES

After completing this chapter, students will be able to:

1. Define *decision model* and describe the importance of such models.
2. Understand the two types of decision models: deterministic and probabilistic models.
3. Understand the steps involved in developing decision models in practical situations.
4. Understand the use of spreadsheets in developing decision models.
5. Discuss possible problems in developing decision models.

CHAPTER OUTLINE

1.1 What Is Decision Modeling?

Decision modeling *is a scientific approach to decision making.*

Although there are several definitions of **decision modeling**, we define it here as a scientific approach to managerial decision making. Alternatively, we can define it as the development of a **model** (usually mathematical) of a real-world problem scenario or environment. The resulting model should typically be such that the decision-making process is not affected by personal bias, whim, emotions, and guesswork. This model can then be used to provide insights into the solution of the managerial problem. Decision modeling is also commonly referred to as *quantitative analysis*, *management science*, or *operations research*. In this textbook, we prefer the term *decision modeling* because we will discuss all modeling techniques in a managerial decision-making context.

Organizations such as American Airlines, United Airlines, IBM, Google, UPS, FedEx, and AT&T frequently use decision modeling to help solve complex problems. Although mathematical tools have been in existence for thousands of years, the formal study and application of quantitative (or mathematical) decision modeling techniques to practical decision making is largely a product of the twentieth century. The decision modeling techniques studied here have been applied successfully to an increasingly wide variety of complex problems in business, government, health care, education, and many other areas. Many such successful uses are discussed throughout this textbook.

It isn't enough, though, just to know the mathematical details of how a particular decision modeling technique can be set up and solved. It is equally important to be familiar with the limitations, assumptions, and specific applicability of the model. The correct use of decision modeling techniques usually results in solutions that are timely, accurate, flexible, economical, reliable, easy to understand, and easy to use.

1.2 Types of Decision Models

Decision models can be broadly classified into two categories, based on the type and nature of the decision-making problem environment under consideration: (1) deterministic models and (2) probabilistic models. We define each of these types of models in the following sections.

Deterministic Models

Deterministic *means with complete certainty.*

Deterministic models assume that all the relevant input data values are known with certainty; that is, they assume that all the information needed for modeling a decision-making problem environment is available, with fixed and known values. An example of such a model is the case of Dell Corporation, which makes several different types of PC products (e.g., desktops, laptops), all of which compete for the same resources (e.g., labor, hard disks, chips, working capital). Dell knows the specific amounts of each resource required to make one unit of each type of PC, based on the PC's design specifications. Further, based on the expected selling price and cost prices of various resources, Dell knows the expected profit contribution per unit of

HISTORY The Origins of Decision Modeling

Decision modeling has been in existence since the beginning of recorded history, but it was Frederick W. Taylor who, in the early 1900s, pioneered the principles of the scientific approach to management. During World War II, many new scientific and quantitative techniques were developed to assist the military. These new developments were so successful that after World War II, many companies started using similar techniques in managerial decision making and planning. Today, many organizations employ a staff of operations research or management science personnel or consultants to apply the principles of scientific management to problems and opportunities. The terms *management science*, *operations research*, and *quantitative analysis* can be used interchangeably, though here we use *decision modeling*.

The origins of many of the techniques discussed in this textbook can be traced to individuals and organizations that have applied the principles of scientific management first developed by Taylor; they are discussed in *History* boxes scattered throughout the textbook.

each type of PC. In such an environment, if Dell decides on a specific production plan, it is a simple task to compute the quantity required of each resource to satisfy that production plan. For example, if Dell plans to ship 50,000 units of a specific laptop model, and each unit includes a pair of 2.0GB SDRAM memory chips, then Dell will need 100,000 units of these memory chips. Likewise, it is easy to compute the total profit that will be realized by this production plan (assuming that Dell can sell all the laptops it makes).

The most commonly used deterministic modeling technique is linear programming.

Perhaps the most common and popular deterministic modeling technique is linear programming (LP). In Chapter 2, we first discuss how small LP models can be set up and solved. We extend our discussion of LP in Chapter 3 to more complex problems drawn from a variety of business disciplines. In Chapter 4, we study how the solution to LP models produces, as a by-product, a great deal of information that is useful for managerial interpretation of the results. Finally, in Chapters 5 and 6, we study a few extensions to LP models. These include several different network flow models (Chapter 5), as well as integer, nonlinear, and multi-objective (goal) programming models (Chapter 6).

As we demonstrate during our study of deterministic models, a variety of important managerial decision-making problems can be set up and solved using these techniques.

Probabilistic Models

Some input data are unknown in probabilistic models.

In contrast to deterministic models, **probabilistic models** (also called *stochastic models*) assume that some *input data* values are not known with certainty. That is, they assume that the values of some important variables will not be known *before* decisions are made. It is therefore important to incorporate this "ignorance" into the model. An example of this type of model is the decision of whether to start a new business venture. As we have seen with the high variability in the stock market during the past several years, the success of such ventures is unsure. However, investors (e.g., venture capitalists, founders) have to make decisions regarding this type of venture, based on their expectations of future performance. Clearly, such expectations are not guaranteed to occur. In recent years, we have seen several examples of firms that have yielded (or are likely to yield) great rewards to their investors (e.g., Google, Facebook, Twitter) and others that have either failed (e.g., eToys.com, Pets.com) or been much more modest in their returns.

Another example of probabilistic modeling to which students may be able to relate easily is their choice of a major when they enter college. Clearly, there is a great deal of uncertainty regarding several issues in this decision-making problem: the student's aptitude for a specific major, his or her actual performance in that major, the employment situation in that major in four years, etc. Nevertheless, a student must choose a major early in his or her college career. Recollect your own situation. In all likelihood, you used your own assumptions (or expectations) regarding the future to evaluate the various alternatives (i.e., you developed a "model" of the decision-making problem). These assumptions may have been the result of information from various sources, such as parents, friends, and guidance counselors. The important point to note here is that none of this information is guaranteed, and no one can predict with 100% accuracy what exactly will happen in the future. Therefore, decisions made with this information, while well thought out and well intentioned, may still turn out to not be the best choices. For example, how many of your friends have changed majors during their college careers?

Because their results are not guaranteed, does this mean that probabilistic decision models are of limited value? As we will see later in this textbook, the answer is an emphatic no. Probabilistic modeling techniques provide a structured approach for managers to incorporate uncertainty into their models and to evaluate decisions under alternate expectations regarding this uncertainty. They do so by using probabilities on the "random," or unknown, variables. Probabilistic modeling techniques discussed in this textbook include decision analysis (Chapter 8), queuing (Chapter 9), simulation (Chapter 10), and forecasting (Chapter 11). Two other techniques, project management (Chapter 7) and inventory control (Chapter 12), include aspects of both deterministic and probabilistic modeling. For each modeling technique, we discuss what kinds of criteria can be used when there is uncertainty and how to use these models to identify the preferred decisions.

Probabilistic models use probabilities to incorporate uncertainty.

Because uncertainty plays a vital role in probabilistic models, some knowledge of basic probability and statistical concepts is useful. Appendix A provides a brief overview of this topic. It should serve as a good refresher while studying these modeling techniques.

Quantitative versus Qualitative Data

The decision modeling process starts with data.

Any decision modeling process starts with data. Like raw material for a factory, these data are manipulated or processed into information that is valuable to people making decisions. This processing and manipulating of raw data into meaningful information is the heart of decision modeling.

Both qualitative and quantitative factors must be considered.

In dealing with a decision-making problem, managers may have to consider both qualitative and quantitative factors. For example, suppose we are considering several different investment alternatives, such as certificates of deposit, the stock market, and real estate. We can use *quantitative* factors such as rates of return, financial ratios, and cash flows in our decision model to guide our ultimate decision. In addition to these factors, however, we may also wish to consider *qualitative* factors such as pending state and federal legislation, new technological breakthroughs, and the outcome of an upcoming election. It can be difficult to quantify these qualitative factors.

Due to the presence (and relative importance) of qualitative factors, the role of quantitative decision modeling in the decision-making process can vary. When there is a lack of qualitative factors, and when the problem, model, and input data remain reasonably stable and steady over time, the results of a decision model can automate the decision-making process. For example, some companies use quantitative inventory models to determine automatically when to order additional new materials and how much to order. In most cases, however, decision modeling is an aid to the decision-making process. The results of decision modeling should be combined with other (qualitative) information while making decisions in practice.

Using Spreadsheets in Decision Modeling

Spreadsheet packages are capable of handling many decision modeling techniques.

In keeping with the ever-increasing presence of technology in modern times, computers have become an integral part of the decision modeling process in today's business environments. Until the early 1990s, many of the modeling techniques discussed here required specialized software packages in order to be solved using a computer. However, spreadsheet packages such as Microsoft Excel have become increasingly capable of setting up and solving most of the decision modeling techniques commonly used in practical situations. For this reason, the current trend in many college courses on decision modeling focuses on spreadsheet-based instruction. In keeping with this trend, we discuss the role and use of spreadsheets (specifically Microsoft Excel) during our study of the different decision modeling techniques presented here.

Several add-ins for Excel are included on the Companion Website for this textbook, www.pearsonhighered.com/balakrishnan.

In addition to discussing the use of some of Excel's built-in functions and procedures (e.g., Goal Seek, Data Table, Chart Wizard), we also discuss several add-ins for Excel. The Data Analysis and Solver add-ins come standard with Excel; others are included on the Companion Website. Table 1.1 lists the add-ins included on the Companion Website and indicates the chapter(s) and topic(s) in which each one is discussed and used.

Because a knowledge of basic Excel commands and procedures facilitates understanding the techniques and concepts discussed here, we recommend reading Appendix B, which provides a brief overview of the Excel features that are most useful in decision modeling. In addition, at appropriate places throughout this textbook, we discuss several Excel functions and procedures specific to each decision modeling technique.

IN ACTION — IBM Uses Decision Modeling to Improve the Productivity of Its Sales Force

IBM is a well-known multinational computer technology, software, and services company with over 380,000 employees and revenue of over $100 billion. A majority of IBM's revenue comes from services, including outsourcing, consulting, and systems integration. At the end of 2007, IBM had approximately 40,000 employees in sales-related roles.

Recognizing that improving the efficiency and productivity of this large sales force can be an effective operational strategy to drive revenue growth and manage expenses, IBM Research developed two broad decision modeling initiatives to facilitate this issue. The first initiative provides a set of analytical models designed to identify new sales opportunities at existing IBM accounts and at noncustomer companies. The second initiative allocates sales resources optimally based on field-validated analytical estimates of future revenue opportunities in market segments. IBM estimates the revenue impact of these two initiatives to be in the several hundreds of millions of dollars each year.

Source: Based on R. Lawrence et al. "Operations Research Improves Sales Force Productivity at IBM," *Interfaces* 40, 1 (January-February 2010): 33–46.

TABLE 1.1
Excel Add-ins Included on This Textbook's Companion Website

EXCEL ADD-IN	USED IN	TOPIC(S)
Tree Plan	Chapter 8	Decision Analysis
Crystal Ball	Chapter 10	Simulation Models
ExcelModules (custom software provided with this textbook)	Chapters 9, 11, and 12	Queuing Models, Forecasting Models, and Inventory Control Models

1.3 Steps Involved in Decision Modeling

The decision modeling process involves three steps.

Regardless of the size and complexity of the decision-making problem at hand, the decision modeling process involves three distinct steps: (1) formulation, (2) solution, and (3) interpretation. Figure 1.1 provides a schematic overview of these steps, along with the components, or parts, of each step. We discuss each of these steps in the following sections.

It is common to iterate between the three steps.

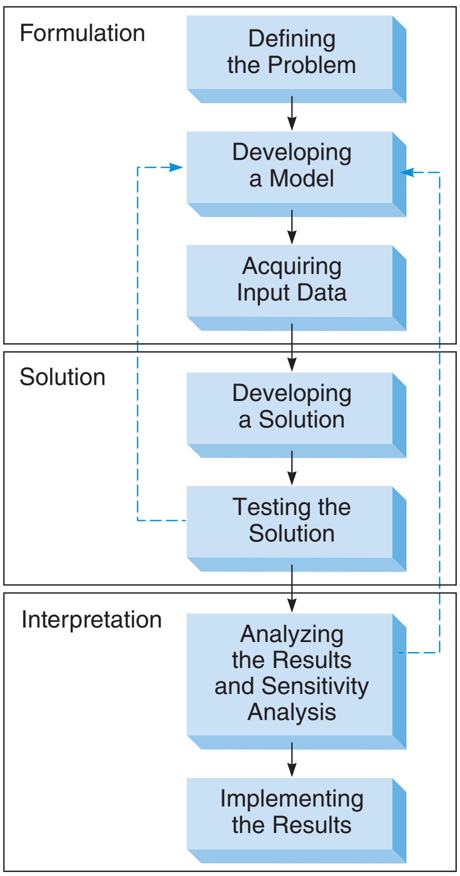
It is important to note that it is common to have an iterative process between these three steps before the final solution is obtained. For example, testing the solution (see Figure 1.1) might reveal that the model is incomplete or that some of the input data are being measured incorrectly. This means that the formulation needs to be revised. This, in turn, causes all the subsequent steps to be changed.

Step 1: Formulation

Formulation *is the most challenging step in decision modeling.*

Formulation is the process by which each aspect of a problem scenario is translated and expressed in terms of a mathematical model. This is perhaps the most important and challenging step in decision modeling because the results of a poorly formulated problem will almost surely be incorrect. It is also in this step that the decision maker's ability to analyze a problem rationally comes into play. Even the most sophisticated software program will not automatically formulate a problem. The aim in formulation is to ensure that the mathematical model completely

FIGURE 1.1
The Decision Modeling Approach



addresses all the issues relevant to the problem at hand. Formulation can be further classified into three parts: (1) defining the problem, (2) developing a model, and (3) acquiring input data.

DEFINING THE PROBLEM The first part in formulation (and in decision modeling) is to develop a clear, concise statement of the problem. This statement gives direction and meaning to all the parts that follow it.

Defining the problem can be the most important part of formulation.

In many cases, defining the problem is perhaps the most important, and the most difficult, part. It is essential to go beyond just the symptoms of the problem at hand and identify the true causes behind it. One problem may be related to other problems, and solving a problem without regard to its related problems may actually make the situation worse. Thus, it is important to analyze how the solution to one problem affects other problems or the decision-making environment in general. Experience has shown that poor problem definition is a major reason for failure of management science groups to serve their organizations well.

When a problem is difficult to quantify, it may be necessary to develop *specific*, *measurable* objectives. For example, say a problem is defined as inadequate health care delivery in a hospital. The objectives might be to increase the number of beds, reduce the average number of days a patient spends in the hospital, increase the physician-to-patient ratio, and so on. When objectives are used, however, the real problem should be kept in mind. It is important to avoid obtaining specific and measurable objectives that may not solve the real problem.

DEVELOPING A MODEL Once we select the problem to be analyzed, the next part is to develop a decision model. Even though you might not be aware of it, you have been using models most of your life. For example, you may have developed the following model about friendship: Friendship is based on reciprocity, an exchange of favors. Hence, if you need a favor, such as a small loan, your model would suggest that you ask a friend.

The types of models include physical, scale, schematic, and mathematical models.

Of course, there are many other types of models. An architect may make a physical model of a building he or she plans to construct. Engineers develop scale models of chemical plants, called pilot plants. A schematic model is a picture or drawing of reality. Automobiles, lawn mowers, circuit boards, typewriters, and numerous other devices have schematic models (drawings and pictures) that reveal how these devices work.

What sets decision modeling apart from other modeling techniques is that the models we develop here are mathematical. A *mathematical model* is a set of mathematical relationships. In most cases, these relationships are expressed as equations and inequalities, as they are in a spreadsheet model that computes sums, averages, or standard deviations.

A **variable** *is a measurable quantity that is subject to change.*

A **parameter** *is a measurable quantity that usually has a known value.*

Although there is considerable flexibility in the development of models, most of the models presented here contain one or more variables and parameters. A **variable**, as the name implies, is a measurable quantity that may vary or that is subject to change. Variables can be controllable or uncontrollable. A controllable variable is also called a *decision variable*. An example is how many inventory items to order. A **problem parameter** is a measurable quantity that is inherent in the problem, such as the cost of placing an order for more inventory items. In most cases, variables are unknown quantities, whereas parameters (or input data) are known quantities.

All models should be developed carefully. They should be solvable, realistic, and easy to understand and modify, and the required input data should be obtainable. A model developer has to be careful to include the appropriate amount of detail for the model to be solvable yet realistic.

ACQUIRING INPUT DATA Once we have developed a model, we must obtain the **input data** to be used in the model. Obtaining accurate data is essential because even if the model is a perfect representation of reality, improper data will result in misleading results. This situation is called *garbage in, garbage out* (GIGO). For larger problems, collecting accurate data can be one of the most difficult aspects of decision modeling.

Garbage in, garbage out means that improper data will result in misleading results.

Several sources can be used in collecting data. In some cases, company reports and documents can be used to obtain the necessary data. Another source is interviews with employees or other persons related to the firm. These individuals can sometimes provide excellent information, and their experience and judgment can be invaluable. A production supervisor, for example, might be able to tell you with a great degree of accuracy the amount of time that it takes to manufacture a particular product. Sampling and direct measurement provide other sources of data for the model. You may need to know how many pounds of a raw material are used in producing a new photochemical product. This information can be obtained by going to the plant and actually measuring the amount of raw material that is being used. In other cases, statistical sampling procedures can be used to obtain data.

Step 2: Solution

In the solution step, we solve the mathematical expressions in the formulation.

The solution step is when the mathematical expressions resulting from the formulation process are actually solved to identify the optimal solution. Until the mid-1990s, typical courses in decision modeling focused a significant portion of their attention on this step because it was the most difficult aspect of studying the modeling process. As stated earlier, thanks to computer technology, the focus today has shifted away from the detailed steps of the solution process and toward the availability and use of software packages. The solution step can be further classified into two parts: (1) developing a solution and (2) testing the solution.

An **algorithm** *is a series of steps that are repeated.*

The input data and model determine the accuracy of the solution.

DEVELOPING A SOLUTION Developing a solution involves manipulating the model to arrive at the best (or optimal) solution to the problem. In some cases, this may require that a set of mathematical expressions be solved to determine the best decision. In other cases, you can use a trial-and-error method, trying various approaches and picking the one that results in the best decision. For some problems, you may wish to try all possible values for the variables in the model to arrive at the best decision; this is called *complete enumeration*. For problems that are quite complex and difficult, you may be able to use an algorithm. An *algorithm* consists of a series of steps or procedures that we repeat until we find the best solution. Regardless of the approach used, the accuracy of the solution depends on the accuracy of the input data and the decision model itself.

Analysts test the data and model before analyzing the results.

TESTING THE SOLUTION Before a solution can be analyzed and implemented, it must be tested completely. Because the solution depends on the input data and the model, both require testing. There are several ways to test input data. One is to collect additional data from a different source and use statistical tests to compare these new data with the original data. If there are significant differences, more effort is required to obtain accurate input data. If the data are accurate but the results are inconsistent with the problem, the model itself may not be appropriate. In this case, the model should be checked to make sure that it is logical and represents the real situation.

Step 3: Interpretation and Sensitivity Analysis

Assuming that the formulation is correct and has been successfully implemented and solved, how does a manager use the results? Here again, the decision maker's expertise is called upon because it is up to him or her to recognize the implications of the results that are presented. We discuss this step in two parts: (1) analyzing the results and sensitivity analysis and (2) implementing the results.

ANALYZING THE RESULTS AND SENSITIVITY ANALYSIS Analyzing the results starts with determining the implications of the solution. In most cases, a solution to a problem will result in some kind of action or change in the way an organization is operating. The implications of these actions or changes must be determined and analyzed before the results are implemented.

Sensitivity analysis *determines how the solutions will change with a different model or input data.*

Because a model is only an approximation of reality, the sensitivity of the solution to changes in the model and input data is an important part of analyzing the results. This type of analysis is called sensitivity, postoptimality, or what-if analysis. **Sensitivity analysis** is used to determine how much the solution will change if there are changes in the model or the input data. When the optimal solution is very sensitive to changes in the input data and the model specifications, additional testing must be performed to make sure the model and input data are accurate and valid.

The importance of sensitivity analysis cannot be overemphasized. Because input data may not always be accurate or model assumptions may not be completely appropriate, sensitivity analysis can become an important part of decision modeling.

IMPLEMENTING THE RESULTS The final part of interpretation is to *implement* the results. This can be much more difficult than one might imagine. Even if the optimal solution will result in millions of dollars in additional profits, if managers resist the new solution, the model is of no value. Experience has shown that a large number of decision modeling teams have failed in their efforts because they have failed to implement a good, workable solution properly.

The solution should be closely monitored even after implementation.

After the solution has been implemented, it should be closely monitored. Over time, there may be numerous changes that call for modifications of the original solution. A changing economy, fluctuating demand, and model enhancements requested by managers and decision makers are a few examples of changes that might require an analysis to be modified.

1.4 Spreadsheet Example of a Decision Model: Tax Computation

A decision modeling example.

Now that we have discussed what a decision model is, let us develop a simple model for a real-world situation that we all face each year: paying taxes. Sue and Robert Miller, a newly married couple, will be filing a joint tax return for the first time this year. Because both work as independent contractors (Sue is an interior decorator, and Rob is a painter), their projected income is subject to some variability. However, because their earnings are not taxed at the source, they know that they have to pay estimated income taxes on a quarterly basis, based on their estimated taxable income for the year. To help calculate this tax, the Millers would like to set up a spreadsheet-based decision model. Assume that they have the following information available:

- Their only source of income is from their jobs.
- They would like to put away 5% of their total income in a retirement account, up to a maximum of $6,000. Any amount they put in that account can be deducted from their total income for tax purposes.
- They are entitled to a personal exemption of $3,700 each. This means that they can deduct $7,400 (= 2 × $3,700) from their total income for tax purposes.
- The standard deduction for married couples filing taxes jointly this year is $11,600. This means that $11,600 of their income is free from any taxes and can be deducted from their total income.
- They do not anticipate having any other deductions from their income for tax purposes.
- The tax brackets for this year are 10% for the first $17,000 of taxable income, 15% between $17,001 and $69,000 and 25% between $69,001 and $139,350. The Millers don't believe that tax brackets beyond $139,350 are relevant for them this year.

Excel Notes

- The Companion Website for this textbook, at www.pearsonhighered.com/balakrishnan, contains the Excel file for each sample problem discussed here. The relevant file name is shown in the margin next to each example.
- In each of our Excel layouts, for clarity, we color code the cells as follows:
  - Variable input cells, in which we enter specific values for the variables in the problem, are shaded yellow.
  - Output cells, which show the results of our analysis, are shaded green.
- We have used callouts to annotate the screenshots in this textbook to highlight important issues in the decision model.
- Wherever necessary, many of these callouts are also included as comments in the Excel files themselves, making it easier for you to understand the logic behind each model.

WWW

File: 1-1.xls, sheet: 1-1A

Screenshot 1-1A shows the formulas that we can use to develop a decision model for the Millers. Just as we have done for this Excel model (and all other models in this textbook), we strongly recommend that you get in the habit of using descriptive titles, labels, and comments in any decision model that you create. The reason for this is very simple: In many real-world settings, decision models that you create are likely to be passed on to others. In such cases, the use of comments will help them understand your thought process. Perhaps an appropriate question you should always ask yourself is "Will I understand this model a year or two after I first write it?" If appropriate labels and comments are included in the model, the answer should always be yes.

Wherever possible, titles, labels, and comments should be included in the model to make them easier to understand.

In Screenshot 1-1A, the known problem parameter values (i.e., constants) are shown in the box labeled Known Parameters. Rather than use these known constant values directly in the formulas, we recommend that you develop the habit of entering each known value in a cell and then using that cell reference in the formulas. In addition to being more "elegant," this way of modeling has the advantage of making any future changes to these values easy.

Rather than use constants directly in formulas, it is preferable to make them cell references.

SCREENSHOT 1-1A
Formula View of Excel Layout for the Millers' Tax Computation

	A	B	C	D	E
1	**Millers' Tax Computation**				
2					
3	**Known Parameters**				
4	Retirement Savings %	0.05			
5	Maximum savings	6000			
6	Personal exemption	3700	per person		
7	Standard deduction	11600			
8	Tax rates	0.1	1	to	17000
9		0.15	17001	to	69000
10		0.25	69001	to	139350
11					
12	**Variables**				
13	Sue's estimated income				
14	Rob's estimated income				
15					
16	**Tax Computation**				
17	Total income	=B13+B14			
18	Retirement savings	=MIN(B4*B17,B5)			
19	Personal exemptions	=2*B6			
20	Standard deduction	=B7			
21	Taxable income	=MAX(0,B17-SUM(B18:B20))			
22	Tax @ 10% rate	=B8*MIN(B21,E8)			
23	Tax @ 15% rate	=IF(B21>E8,B9*(MIN(B21,E9)-E8),0)			
24	Tax @ 25% rate	=IF(B21>E9,B10*(MIN(B21,E10)-E9),0)			
25	Total tax	=SUM(B22:B24)			
26	Estimated tax per quarter	=B25/4			

This box shows all the *known* input parameter values.

This box shows the two input variables.

Minimum of (5% of total income, $6,000)

Maximum of (0, taxable income)

10% tax up to $17,000

15% tax between $17,001 and $69,000. This tax is calculated only if taxable income exceeds $17,000.

25% tax between $69,001 and $139,350. This tax is calculated only if taxable income exceeds $69,000.

Cells B13 and B14 denote the only two variable data entries in this decision model: Sue's and Rob's estimated incomes for this year. When we enter values for these two variables, the results are computed in cells B17:B26 and presented in the box labeled Tax Computation.

Excel's MAX, MIN, and IF functions have been used in this decision model.

Cell B17 shows the total income. The MIN function is used in cell B18 to specify the tax-deductible retirement contribution as the smaller value of 5% of total income and $6,000. Cells B19 and B20 set the personal exemptions and the standard deduction, respectively. The net taxable income is shown in cell B21, and the MAX function is used here to ensure that this amount is never below zero. The taxes payable at the 10%, 15%, and 25% rates are then calculated in cells B22, B23, and B24, respectively. In each of these cells, the MIN function is used to ensure that only the incremental taxable income is taxed at a given rate. (For example, in cell B23, only the portion of taxable income above $17,000 is taxed at the 15% rate, up to an upper limit of $69,000.) The IF function is used in cells B23 and B24 to check whether the taxable income exceeds the lower limit for the 15% and 25% tax rates, respectively. If the taxable income does not exceed the relevant lower limit, the IF function sets the tax payable at that rate to zero. Finally, the total tax payable is computed in cell B25, and the estimated quarterly tax is computed in cell B26.

WWW

File: 1-1.xls, sheet: 1-1B

Now that we have developed this decision model, how can the Millers actually use it? Suppose Sue estimates her income this year at $55,000 and Rob estimates his at $50,000. We enter these values in cells B13 and B14, respectively. The decision model immediately lets us know that the Millers have a taxable income of $80,750 and that they should pay estimated taxes of $3,109.38 each quarter. These input values, and the resulting computations, are shown in Screenshot 1-1B. We can use this decision model in a similar fashion with any other estimated income values for Sue and Rob.

Observe that the decision model we have developed for the Millers' example does not optimize the decision in any way. That is, the model simply computes the estimated taxes for a given income level. It does not, for example, determine whether these taxes can be reduced in some way through better tax planning. Later in this textbook, we discuss decision models that not only help compute the implications of a particular specified decision but also help identify the optimal decision, based on some objective or goal.

SCREENSHOT 1-1B
Excel Decision Model for the Millers' Tax Computation

	A	B	C	D	E
1	**Millers' Tax Computation**				
2					
3	**Known Parameters**				
4	Retirement Savings %	5.0%			
5	Maximum savings	$6,000			
6	Personal exemption	$3,700	per person		
7	Standard deduction	$11,600			
8	Tax rates	10.0%	$1	to	$17,000
9		15.0%	$17,001	to	$69,000
10		25.0%	$69,001	to	$139,350
11					
12	**Variables**				
13	Sue's estimated income	$55,000.00			
14	Rob's estimated income	$50,000.00			
15					
16	**Tax Computation**				
17	Total income	$105,000.00			
18	Retirement savings	$5,250.00			
19	Personal exemptions	$7,400.00			
20	Standard deduction	$11,600.00			
21	Taxable income	$80,750.00			
22	Tax @ 10% rate	$1,700.00			
23	Tax @ 15% rate	$7,800.00			
24	Tax @ 25% rate	$2,937.50			
25	Total tax	$12,437.50			
26	Estimated tax per quarter	$3,109.38			

Estimated income

Total income of $105,000 has been reduced to taxable income of only $80,750.

The Millers should pay $3,109.38 in estimated taxes each quarter.

IN ACTION Using Decision Modeling to Combat Spread of Hepatitis B Virus in the United States and China

Hepatitis B is a vaccine-preventable viral disease that is a major public health problem, particularly among Asian populations. Left untreated, it can lead to death from cirrhosis and liver cancer. Over 350 million people are chronically infected with the hepatitis B virus (HBV) worldwide. In the United States (US), although about 10% of Asian and Pacific Islanders are chronically infected, about two thirds of them are unaware of their infection. In China, HBV infection is a leading cause of death.

During several years of work conducted at the Asian Liver Center at Stanford University, the authors used combinations of decision modeling techniques to analyze the cost effectiveness of various intervention schemes to combat the spread of the disease in the US and China. The results of these analyses have helped change US public health policy on hepatitis B screening, and have helped encourage China to enact legislation to provide free vaccination for millions of children.

These policies are an important step in eliminating health disparities and ensuring that millions of people can now receive the hepatitis B vaccination they need. The Global Health Coordinator of the Asian Liver Center states that this research "has been incredibly important to accelerating policy changes to improve health related to HBV."

Source: Based on D. W. Hutton, M. L. Brandeau, and S. K. So. "Doing Good with Good OR: Supporting Cost-Effective Hepatitis B Interventions," *Interfaces* 41, 3 (May-June 2011): 289–300.

1.5 Spreadsheet Example of a Decision Model: Break-Even Analysis

Expenses include fixed and variable costs.

Let us now develop another decision model—this one to compute the total profit for a firm as well as the associated break-even point. We know that profit is simply the difference between revenue and expense. In most cases, we can express revenue as the selling price per unit multiplied by the number of units sold. Likewise, we can express expense as the sum of the total fixed and variable costs. In turn, the total variable cost is the variable cost per unit multiplied by the number of units sold. Thus, we can express profit using the following mathematical expression:

$$\begin{aligned}\text{Profit} &= (\text{Selling price per unit}) \times (\text{Number of units}) - (\text{Fixed cost}) \\ &\quad - (\text{Variable cost per unit}) \times (\text{Number of units}) \qquad \text{(1-1)}\end{aligned}$$

Let's use Bill Pritchett's clock repair shop as an example to demonstrate the creation of a decision model to calculate profit and the associated break-even point. Bill's company, Pritchett's Precious Time Pieces, buys, sells, and repairs old clocks and clock parts. Bill sells rebuilt springs for a unit price of \$10. The fixed cost of the equipment to build the springs is \$1,000. The variable cost per unit is \$5 for spring material. If we represent the number of springs (units) sold as the variable X, we can restate the profit as follows:

$$\text{Profit} = \$10X - \$1{,}000 - \$5X$$

File: 1-2.xls, sheets: 1-2A and 1-2B

Screenshot 1-2A shows the formulas used in developing the decision model for Bill Pritchett's example. Cells B4, B5, and B6 show the known problem parameter values—namely, revenue per unit, fixed cost, and variable cost per unit, respectively. Cell B9 is the lone variable in the model, and it represents the number of units sold (i.e., X). Using these entries, the total revenue, total variable cost, total cost, and profit are computed in cells B12, B14, B15, and B16, respectively. For example, if we enter a value of 1,000 units for X in cell B9, the profit is calculated as \$4,000 in cell B16, as shown in Screenshot 1-2B.

The BEP results in \$0 profit.

In addition to computing the profit, decision makers are often interested in the **break-even point (BEP)**. The BEP is the number of units sold that will result in total revenue equaling total costs (i.e., profit is \$0). We can determine the BEP analytically by setting profit equal to \$0 and solving for X in Bill Pritchett's profit expression. That is

SCREENSHOT 1-2A
Formula View of Excel Layout for Pritchett's Precious Time Pieces

	A	B
1	**Bill Pritchett's Shop**	
2		
3	**Known Parameters**	
4	Selling price per unit	10
5	Fixed cost	1000
6	Variable cost per unit	5
7		
8	**Variables**	
9	Number of units, X	
10		
11	**Results**	
12	Total revenue	=B4*B9
13	Fixed cost	=B5
14	Total variable cost	=B6*B9
15	Total cost	=B13+B14
16	Profit	=B12-B15

Input variable

Profit is revenue – fixed cost – variable cost.

SCREENSHOT 1-2B
Excel Decision Model for Pritchett's Precious Time Pieces

	A	B
1	**Bill Pritchett's Shop**	
2		
3	**Known Parameters**	
4	Selling price per unit	\$10.00
5	Fixed cost	\$1,000.00
6	Variable cost per unit	\$5.00
7		
8	**Variables**	
9	Number of units, X	1000
10		
11	**Results**	
12	Total revenue	\$10,000.00
13	Fixed cost	\$1,000.00
14	Total variable cost	\$5,000.00
15	Total cost	\$6,000.00
16	Profit	\$4,000.00

1,000 units sold

Profit is \$4,000 if 1,000 units are sold.

$$0 = (\text{Selling price per unit}) \times (\text{Number of units}) - (\text{Fixed cost}) - (\text{Variable cost per unit}) \times (\text{Number of units})$$

which can be mathematically rewritten as

$$\text{Break even point (BEP)} = \text{Fixed cost}/(\text{Selling price per unit} - \text{Variable cost per unit}) \quad \textbf{(1-2)}$$

For Bill Pritchett's example, we can compute the BEP as $1,000/($10 – $5) = 200 springs. The **BEP in dollars** (which we denote as $BEP_\$$) can then be computed as

$$BEP_\$ = \text{Fixed cost} + \text{Variable costs} \times \text{BEP} \quad \textbf{(1-3)}$$

For Bill Pritchett's example, we can compute $BEP_\$$ as $1,000 + $5 × 200 = $2,000.

Using Goal Seek to Find the Break-Even Point

Excel's **Goal Seek** *can be used to automatically find the BEP.*

While the preceding analytical computations for BEP and $BEP_\$$ are fairly simple, an advantage of using computer-based models is that many of these results can be calculated automatically. For example, we can use a procedure in Excel called **Goal Seek** to calculate the BEP and $BEP_\$$ values in the decision model shown in Screenshot 1-2B. The Goal Seek procedure allows us to specify a desired value for a *target cell*. This target cell should contain a formula that involves a different cell, called the *changing cell*. Once we specify the target cell, its desired value, and the changing cell in Goal Seek, the procedure automatically manipulates the changing cell value to try and make the target cell achieve its desired value.

In our case, we want to manipulate the value of the number of units X (in cell B9 of Screenshot 1-2B) such that the profit (in cell B16 of Screenshot 1-2B) takes on a value of zero. That is, cell B16 is the target cell, its desired value is zero, and cell B9 is the changing cell. Observe that the formula of profit in cell B16 is a function of the value of X in cell B9 (see Screenshot 1-2A).

File: 1-2.xls, sheet: 1-2C

Screenshot 1-2C shows how the Goal Seek procedure is implemented in Excel. As shown in Screenshot 1-2C(a), we invoke Goal Seek by clicking the Data tab on Excel's main menu bar, followed by the What-If Analysis button (found in the Data Tools group within the Data tab), and then finally on Goal Seek. The window shown in Screenshot 1-2C(b) is displayed. We specify cell B16 in the Set cell box, a desired value of zero for this cell in the To value box, and cell B9 in the By changing cell box. When we now click OK, the Goal Seek Status window shown in Screenshot 1-2C(c) is displayed, indicating that the target of $0 profit has been achieved. Cell B9 shows the resulting BEP value of 200 units. The corresponding $BEP_\$$ value of $2,000 is shown in cell B15.

Observe that we can use Goal Seek to compute the sales level needed to obtain any desired profit. For example, see if you can verify that in order to get a profit of $10,000, Bill Pritchett would have to sell 2,200 springs. We will use the Goal Seek procedure again in Chapter 9.

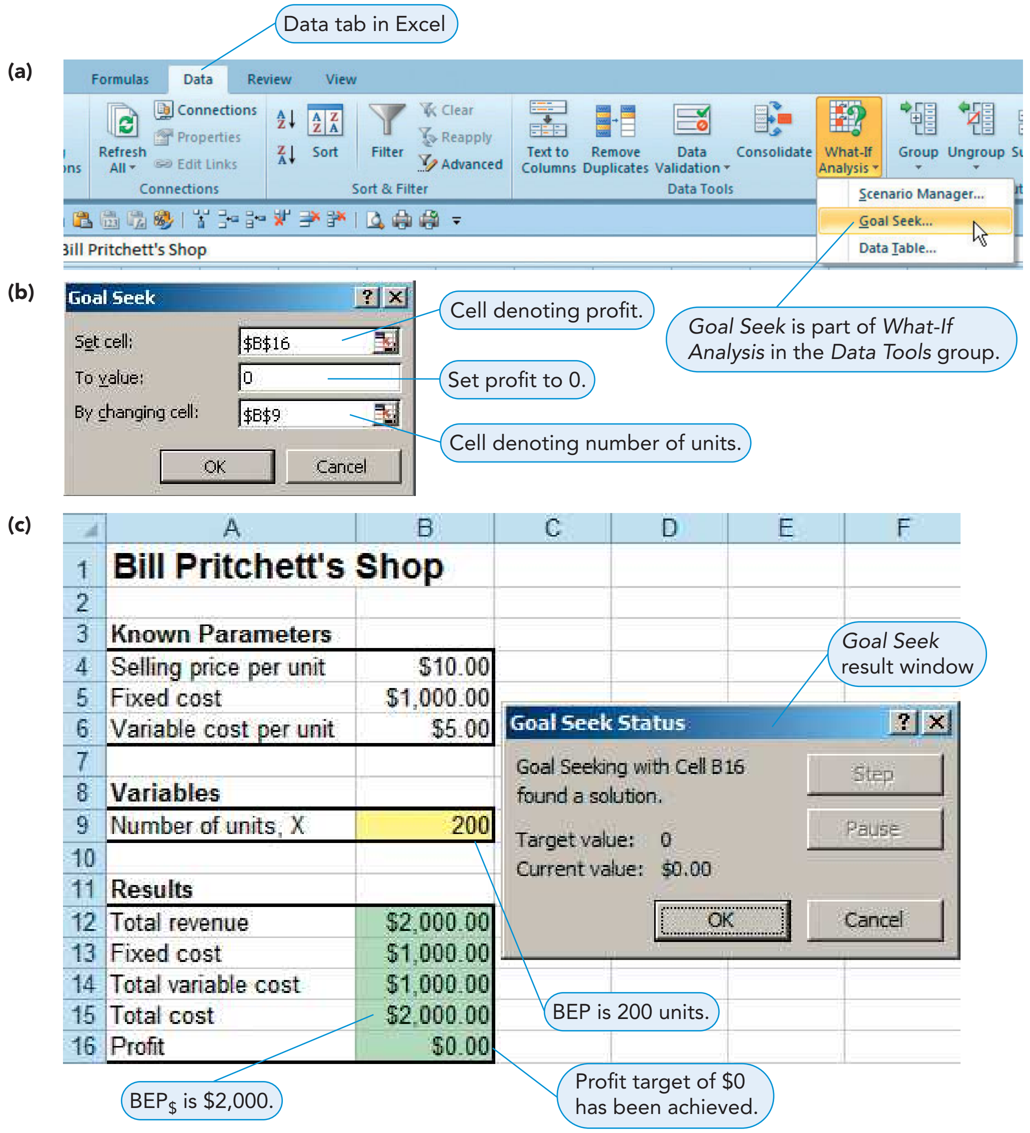
IN ACTION The Management Science Group Is Bullish at Merrill Lynch

Management science groups at corporations can make a huge difference in reducing costs and increasing profits. At Merrill Lynch, the management science group was established in 1986. Its overall mission is to provide high-quality quantitative (or mathematical) analysis, modeling, and decision support. The group analyzes a variety of problems and opportunities related to client services, products, and the marketplace. In the past, this group has helped Merrill Lynch develop asset allocation models, mutual fund portfolio optimization solutions, investment strategy development and research tools, financial planning models, and cross-selling approaches.

To provide meaningful assistance to Merrill Lynch, the management science group has concentrated on mathematical models that focus on client satisfaction. What are the keys to continued success for the management science group? Although skill and technical expertise in decision modeling are essential, the management science group has identified the following four critical success factors: (1) objective analysis, (2) focus on business impact and implementation, (3) teamwork, and (4) adopting a disciplined consultative approach.

Source: From R. Nigam et al. "Bullish on Management Science," *OR/MS Today*, Vol. 27, No. 3 (June 2000): 48-51. Reprinted with permission.

SCREENSHOT 1-2C
Using Excel's Goal Seek to Compute the Break-Even Point for Pritchett's Precious Time Pieces



1.6 Possible Problems in Developing Decision Models

We present the decision modeling approach as a logical and systematic means of tackling decision-making problems. Even when these steps are followed carefully, however, many difficulties can hurt the chances of implementing solutions to real-world problems. We now look at problems that can occur during each of the steps of the decision modeling approach.

Defining the Problem

Real-world problems are not easily identifiable.

In the worlds of business, government, and education, problems are, unfortunately, not easily identified. **Decision analysts** typically face four roadblocks in defining a problem. We use an application, inventory analysis, throughout this section as an example.

The problem needs to be examined from several viewpoints.

CONFLICTING VIEWPOINTS Analysts may often have to consider conflicting viewpoints in defining a problem. For example, in inventory problems, financial managers usually feel that inventory is too high because inventory represents cash not available for other investments. In contrast, sales managers often feel that inventory is too low because high levels of inventory may be needed to fill unexpected orders. If analysts adopt either of these views as the problem definition, they have essentially accepted one manager's perception. They can, therefore, expect resistance from the other manager when the "solution" emerges. So it's important to consider both points of view before stating the problem.

IMPACT ON OTHER DEPARTMENTS Problems do not exist in isolation and are not owned by just one department of a firm. For example, inventory is closely tied with cash flows and various production problems. A change in ordering policy can affect cash flows and upset production schedules to the point that savings on inventory are exceeded by increased financial and production costs. The problem statement should therefore be as broad as possible and include inputs from all concerned departments.

All inputs must be considered.

BEGINNING ASSUMPTIONS People often have a tendency to state problems in terms of solutions. For example, the statement that inventory is too low implies a solution: that its levels should be raised. An analyst who starts off with this assumption will likely find that inventory should be raised! From an implementation perspective, a "good" solution to the right problem is much better than an "optimal" solution to the wrong problem.

SOLUTION OUTDATED Even if a problem has been specified correctly at present, it can change during the development of the model. In today's rapidly changing business environment, especially with the amazing pace of technological advances, it is not unusual for problems to change virtually overnight. The analyst who presents solutions to problems that no longer exist can't expect credit for providing timely help.

Developing a Model

Even with a well-defined problem statement, a decision analyst may have to overcome hurdles while developing decision models for real-world situations. Some of these hurdles are discussed in the following sections.

FITTING THE TEXTBOOK MODELS A manager's perception of a problem does not always match the textbook approach. For example, most textbook inventory models involve minimizing the sum of holding and ordering costs. Some managers view these costs as unimportant; instead, they see the problem in terms of cash flow, turnover, and levels of customer satisfaction. The results of a model based on holding and ordering costs are probably not acceptable to such managers.

Managers do not use the results of a model they do not understand.

UNDERSTANDING A MODEL Most managers simply do not use the results of a model they do not understand. Complex problems, though, require complex models. One trade-off is to simplify assumptions in order to make a model easier to understand. The model loses some of its reality but gains some management acceptance. For example, a popular simplifying assumption in inventory modeling is that demand is known and constant. This allows analysts to build simple, easy-to-understand models. Demand, however, is rarely known and constant, so these models lack some reality. Introducing probability distributions provides more realism but may put comprehension beyond all but the most mathematically sophisticated managers. In such cases, one approach is for the decision analyst to start with the simple model and make sure that it is completely understood. More complex models can then be introduced slowly as managers gain more confidence in using these models.

Acquiring Input Data

Gathering the data to be used in the decision modeling approach to problem solving is often not a simple task. Often, the data are buried in several different databases and documents, making it very difficult for a decision analyst to gain access to the data.

USING ACCOUNTING DATA One problem is that most data generated in a firm come from basic accounting reports. The accounting department collects its inventory data, for example, in terms of cash flows and turnover. But decision analysts tackling an inventory problem need to collect data on holding costs and ordering costs. If they ask for such data, they may be shocked to find that the data were simply never collected for those specified costs.

Professor Gene Woolsey tells a story of a young decision analyst sent down to accounting to get "the inventory holding cost per item per day for part 23456/AZ." The accountant asked the young man if he wanted the first-in, first-out figure; the last-in, first-out figure; the lower of cost or market figure; or the "how-we-do-it" figure. The young man replied that the inventory model required only one number. The accountant at the next desk said, "Heck, Joe, give the kid a number." The analyst was given a number and departed.

The results of a model are only as good as the input data used.

VALIDITY OF DATA A lack of "good, clean data" means that whatever data are available must often be distilled and manipulated (we call it "fudging") before being used in a model. Unfortunately, the validity of the results of a model is no better than the validity of the data that go into the model. You cannot blame a manager for resisting a model's "scientific" results when he or she knows that questionable data were used as input.

Developing a Solution

An analyst may have to face two potential pitfalls while developing solutions to a decision model. These are discussed in the following sections.

HARD-TO-UNDERSTAND MATHEMATICS The first concern in developing solutions is that although the mathematical models we use may be complex and powerful, they may not be completely understood. The aura of mathematics often causes managers to remain silent when they should be critical. The well-known management scientist C. W. Churchman once cautioned that "because mathematics has been so revered a discipline in recent years, it tends to lull the unsuspecting into believing that he who thinks elaborately thinks well."[1]

Hard-to-understand mathematics and having only one answer can be problems in developing a solution.

THE LIMITATION OF ONLY ONE ANSWER The second problem in developing a solution is that decision models usually give just one answer to a problem. Most managers would like to have a range of options and not be put in a take-it-or-leave-it position. A more appropriate strategy is for an analyst to present a range of options, indicating the effect that each solution has on the objective function. This gives managers a choice as well as information on how much it will cost to deviate from the optimal solution. It also allows problems to be viewed from a broader perspective because it means that qualitative factors can also be considered.

Testing the Solution

Assumptions should be reviewed.

The results of decision modeling often take the form of predictions of how things will work in the future if certain changes are made in the present. To get a preview of how well solutions will really work, managers are often asked how good a solution looks to them. The problem is that complex models tend to give solutions that are not intuitively obvious. And such solutions tend to be rejected by managers. Then a decision analyst must work through the model and the assumptions with the manager in an effort to convince the manager of the validity of the results. In the process of convincing the manager, the analyst has to review every assumption that went into the model. If there are errors, they may be revealed during this review. In addition, the manager casts a critical eye on everything that went into the model, and if he or she can be convinced that the model is valid, there is a good chance that the solution results are also valid.

Analyzing the Results

Once a solution has been tested, the results must be analyzed in terms of how they will affect the total organization. You should be aware that even small changes in organizations are often difficult to bring about. If results suggest large changes in organizational policy, the decision analyst can expect resistance. In analyzing the results, the analyst should ascertain who must change and by how much, if the people who must change will be better or worse off, and who has the power to direct the change.

1.7 Implementation—Not Just the Final Step

We have just presented some of the many problems that can affect the ultimate acceptance of decision modeling in practice. It should be clear now that implementation isn't just another step that takes place after the modeling process is over. Each of these steps greatly affects the chances of implementing the results of a decision model.

Even though many business decisions can be made intuitively, based on hunches and experience, there are more and more situations in which decision models can assist. Some managers, however, fear that the use of a formal analytical process will reduce their

[1] Churchman, C. W. "Reliability of Models in the Social Sciences," *Interfaces* 4, 1 (November 1973): 1–12.

decision-making power. Others fear that it may expose some previous intuitive decisions as inadequate. Still others feel uncomfortable about having to reverse their thinking patterns with formal decision making. These managers often argue against the use of decision modeling.

Many action-oriented managers do not like the lengthy formal decision-making process and prefer to get things done quickly. They prefer "quick and dirty" techniques that can yield immediate results. However, once managers see some quick results that have a substantial payoff, the stage is set for convincing them that decision modeling is a beneficial tool.

Management support and user involvement are important.

We have known for some time that management support and user involvement are critical to the successful implementation of decision modeling processes. A Swedish study found that only 40% of projects suggested by decision analysts were ever implemented. But 70% of the modeling projects initiated by users, and fully 98% of projects suggested by top managers, were implemented.

Summary

Decision modeling is a scientific approach to decision making in practical situations faced by managers. Decision models can be broadly classified into two categories, based on the type and nature of the problem environment under consideration: (1) deterministic models and (2) probabilistic models. Deterministic models assume that all the relevant input data and parameters are known with certainty. In contrast, probabilistic models assume that some input data are not known with certainty. The decision modeling approach includes three major steps: (1) formulation, (2) solution, and (3) interpretation. It is important to note that it is common to iterate between these three steps before the final solution is obtained. Spreadsheets are commonly used to develop decision models.

In using the decision modeling approach, however, there can be potential problems, such as conflicting viewpoints, disregard of the impact of the model on other departments, outdated solutions, misunderstanding of the model, difficulty acquiring good input data, and hard-to-understand mathematics. In using decision models, implementation is not the final step. There can be a lack of commitment to the approach and resistance to change.

Glossary

Break-Even Point (BEP) The number of units sold that will result in total revenue equaling total costs (i.e., profit is $0).

Break-Even Point in Dollars ($BEP_\$$) The sum of fixed and total variable cost if the number of units sold equals the break-even point.

Decision Analyst An individual who is responsible for developing a decision model.

Decision Modeling A scientific approach that uses quantitative (mathematical) techniques as a tool in managerial decision making. Also known as *quantitative analysis*, *management science*, and *operations research*.

Deterministic Model A model which assumes that all the relevant input data and parameters are known with certainty.

Formulation The process by which each aspect of a problem scenario is translated and expressed in terms of a mathematical model.

Goal Seek A feature in Excel that allows users to specify a goal or target for a specific cell and automatically manipulate another cell to achieve that target.

Input Data Data that are used in a model in arriving at the final solution.

Model A representation (usually mathematical) of a practical problem scenario or environment.

Probabilistic Model A model which assumes that some input data are not known with certainty.

Problem Parameter A measurable quantity that is inherent in a problem. It typically has a fixed and known value (i.e., a constant).

Sensitivity Analysis A process that involves determining how sensitive a solution is to changes in the formulation of a problem.

Variable A measurable quantity that may vary or that is subject to change.

Discussion Questions and Problems

Discussion Questions

1-1 Define *decision modeling*. What are some of the organizations that support the use of the scientific approach?

1-2 What is the difference between deterministic and probabilistic models? Give several examples of each type of model.

1-3 What are the differences between quantitative and qualitative factors that may be present in a decision model?

1-4 Why might it be difficult to quantify some qualitative factors in developing decision models?

1-5 What steps are involved in the decision modeling process? Give several examples of this process.

1-6 Why is it important to have an iterative process between the steps of the decision modeling approach?

1-7 Briefly trace the history of decision modeling. What happened to the development of decision modeling during World War II?

1-8 What different types of models are mentioned in this chapter? Give examples of each.

1-9 List some sources of input data.

1-10 Define *decision variable*. Give some examples of variables in a decision model.

1-11 What is a problem parameter? Give some examples of parameters in a decision model.

1-12 List some advantages of using spreadsheets for decision modeling.

1-13 What is implementation, and why is it important?

1-14 Describe the use of sensitivity analysis, or postoptimality analysis, in analyzing the results of decision models.

1-15 Managers are quick to claim that decision modelers talk to them in a jargon that does not sound like English. List four terms that might not be understood by a manager. Then explain in nontechnical terms what each of them means.

1-16 Why do you think many decision analysts don't like to participate in the implementation process? What could be done to change this attitude?

1-17 Should people who will be using the results of a new modeling approach become involved in the technical aspects of the problem-solving procedure?

1-18 C. W. Churchman once said that "mathematics tends to lull the unsuspecting into believing that he who thinks elaborately thinks well." Do you think that the best decision models are the ones that are most elaborate and complex mathematically? Why?

Problems

1-19 A Website has a fixed cost $15,000 per day. The revenue is $0.06 each time the Website is accessed. The variable cost of responding to each hit is $0.02.
(a) How many times must this Website be accessed each day to break even?
(b) What is the break-even point, in dollars?

1-20 An electronics firm is currently manufacturing an item that has a variable cost of $0.60 per unit and selling price of $1.10 per unit. Fixed costs are $15,500. Current volume is 32,000 units. The firm can substantially improve the product quality by adding a new piece of equipment at an additional fixed cost of $8,000. Variable cost would increase to $0.70, but volume is expected to jump to 50,000 units due to the higher quality of the product.
(a) Should the company buy the new equipment?
(b) Compute the profit with the current equipment and the expected profit with the new equipment.

1-21 A manufacturer is evaluating options regarding his production equipment. He is trying to decide whether he should refurbish his old equipment for $70,000, make major modifications to the production line for $135,000, or purchase new equipment for $230,000. The product sells for $10, but the variable costs to make the product are expected to vary widely, depending on the decision that is to be made regarding the equipment. If the manufacturer refurbishes, the variable costs will be $7.20 per unit. If he modifies or purchases new equipment, the variable costs are expected to be $5.25 and $4.75, respectively.
(a) Which alternative should the manufacturer choose if it the demand is expected to be between 30,000 and 40,000 units?
(b) What will be the manufacturer's profit if the demand is 38,000 units?

1-22 St. Joseph's School has 1,200 students, each of whom currently pays $8,000 per year to attend. In addition to revenues from tuition, the school receives an appropriation from the church to sustain its activity. The budget for the upcoming year is $15 million, and the church appropriation will be $4.8 million. By how much will the school have to raise tuition per student to keep from having a shortfall in the upcoming year?

1-23 Refer to Problem 1-22. Sensing resistance to the idea of raising tuition from members of St. Joseph's Church, one of the board members suggested that the 960 children of church members could pay $8,000 as usual. Children of nonmembers would pay more. What would the nonmember tuition per year be if St. Joseph's wanted to continue to plan for a $15 million budget?

1-24 Refer to Problems 1-22 and 1-23. Another board member believes that if church members pay $8,000 in tuition, the most St. Joseph's can increase nonmember tuition is $1,000 per year. She suggests that another solution might be to cap nonmember tuition at $9,000 and attempt to recruit more nonmember students to make up the shortfall. Under this plan, how many new nonmember students will need to be recruited?

1-25 Great Lakes Automotive is considering producing, in-house, a gear assembly that it currently purchases from Delta Supply for $6 per unit. Great Lakes estimates that if it chooses to manufacture the gear assembly, it will cost $23,000 to set up the process and then $3.82 per unit for labor and materials. At what volume would these options cost Great Lakes the same amount of money?

1-26 A start-up publishing company estimates that the fixed costs of its first major project will be $190,000, the variable cost will be $18, and the selling price per book will be $34.

(a) How many books must be sold for this project to break even?

(b) Suppose the publishers wish to take a total of $40,000 in salary for this project. How many books must be sold to break even, and what is the break-even point, in dollars?

1-27 The electronics firm in Problem 1-20 is now considering purchasing the new equipment and increasing the selling price of its product to $1.20 per unit. Even with the price increase, the new volume is expected to be 50,000 units. Under these circumstances, should the company purchase the new equipment and increase the selling price?

1-28 A distributer of prewashed shredded lettuce is opening a new plant and considering whether to use a mechanized process or a manual process to prepare the product. The manual process will have a fixed cost of $43,400 per month and a variable cost of $1.80 per 5-pound bag. The mechanized process would have a fixed cost of $84,600 per month and a variable cost of $1.30 per bag. The company expects to sell each bag of shredded lettuce for $2.50.

(a) Find the break-even point for each process.

(b) What is the monthly profit or loss if the company chooses the manual process and sells 70,000 bags per month?

1-29 A fabrication company must replace its widget machine and is evaluating the capabilities of two available machines. Machine A would cost the company $75,000 in fixed costs for the first year. Each widget produced using Machine A would have a variable cost of $16. Machine B would have a first-year fixed cost of $62,000, and widgets made on this machine would have a variable cost of $20. Machine A would have the capacity to make 18,000 widgets per year, which is approximately double the capacity for Machine B.

(a) If widgets sell for $28 each, find the break-even point for each machine. Consider first-year costs only.

(b) If the fabrication company estimates a demand of 6,500 units in the next year, which machine should be selected?

(c) At what level of production do the two production machines cost the same?

1-30 Bismarck Manufacturing intends to increase capacity through the addition of new equipment. Two vendors have presented proposals. The fixed cost for proposal A is $65,000, and for proposal B, $34,000. The variable cost for A is $10, and for B, $14. The revenue generated by each unit is $18.

(a) What is the break-even point for each proposal?

(b) If the expected volume is 8,300 units, which alternative should be chosen?

CHAPTER 2

Linear Programming Models: Graphical and Computer Methods

LEARNING OBJECTIVES

After completing this chapter, students will be able to:

1. Understand the basic assumptions and properties of linear programming (LP).
2. Use graphical procedures to solve LP problems with only two variables to understand how LP problems are solved.
3. Understand special situations such as redundancy, infeasibility, unboundedness, and alternate optimal solutions in LP problems.
4. Understand how to set up LP problems on a spreadsheet and solve them using Excel's Solver.

CHAPTER OUTLINE

2.1 Introduction

Management decisions in many organizations involve trying to make the most effective use of resources. Resources typically include machinery, labor, money, time, warehouse space, and raw materials. These resources can be used to manufacture products (e.g., computers, automobiles, furniture, clothing) or provide services (e.g., package delivery, health services, advertising policies, investment decisions).

In all resource allocation situations, the manager must sift through several thousand decision choices or alternatives to identify the best, or optimal, choice. The most widely used decision modeling technique designed to help managers in this process is called **mathematical programming**. The term *mathematical programming* is somewhat misleading because this modeling technique requires no advanced mathematical ability (it uses just basic algebra) and has nothing whatsoever to do with computer software programming! In the world of decision modeling, *programming* refers to setting up and solving a problem mathematically.

Linear programming helps in resource allocation decisions.

Within the broad topic of mathematical programming, the most widely used modeling technique designed to help managers in planning and decision making is **linear programming (LP)**. We devote this and the next two chapters to illustrating how, why, and where LP works. Then, in Chapter 5, we explore several special LP models called *network flow problems*. We follow that with a discussion of a few other mathematical programming techniques (i.e., integer programming, goal programming, and nonlinear programming) in Chapter 6.

When developing LP (and other mathematical programming)–based decision models, we assume that all the relevant input data and parameters are known with certainty. For this reason, these types of decision modeling techniques are classified as *deterministic* models.

We focus on using Excel to set up and solve LP models.

Computers have, of course, played an important role in the advancement and use of LP. Real-world LP problems are too cumbersome to solve by hand or with a calculator, and computers have become an integral part of setting up and solving LP models in today's business environments. As noted in Chapter 1, over the past decade, spreadsheet packages such as Microsoft Excel have become increasingly capable of handling many of the decision modeling techniques (including LP and other mathematical programming models) that are commonly encountered in practical situations. So throughout the chapters on mathematical programming techniques, we discuss the role and use of Microsoft Excel in setting up and solving these models.

2.2 Developing a Linear Programming Model

Since the mid-twentieth century, LP has been applied extensively to medical, transportation, operations, financial, marketing, accounting, human resources, and agricultural problems. Regardless of the size and complexity of the decision-making problem at hand in these diverse applications, the development of all LP models can be viewed in terms of the three distinct steps, as defined in Chapter 1: (1) formulation, (2) solution, and (3) interpretation. We now discuss each with regard to LP models.

HISTORY How Linear Programming Started

Linear programming was conceptually developed before World War II by the outstanding Soviet mathematician A. N. Kolmogorov. Another Russian, Leonid Kantorovich, won the Nobel Prize in Economics for advancing the concepts of optimal planning. An early application of linear programming, by George Stigler in 1945, was in the area we today call "diet problems."

Major progress in the field, however, took place in 1947 and later, when George D. Dantzig developed the solution procedure known as the *simplex algorithm*. Dantzig, then a U.S. Air Force mathematician, was assigned to work on logistics problems. He noticed that many problems involving limited resources and more than one demand could be set up in terms of a series of equations and inequalities. Although early LP applications were military in nature, industrial applications rapidly became apparent with the spread of business computers. In 1984, Narendra Karmarkar developed an algorithm that appears to be superior to the simplex method for many very large applications.

Formulation

Formulation *involves expressing a problem scenario in terms of simple mathematical expressions.*

Formulation is the process by which each aspect of a problem scenario is translated and expressed in terms of simple mathematical expressions. The aim in LP formulation is to ensure that the set of mathematical equations, taken together, completely addresses all the issues relevant to the problem situation at hand. We demonstrate a few examples of simple LP formulations in this chapter. Then we introduce several more comprehensive formulations in Chapter 3.

Solution

Solution *involves solving mathematical expressions to find values for the variables.*

The *solution* step is where the mathematical expressions resulting from the formulation process are solved to identify *an* optimal (or best) solution to the model.[1] In this textbook, the focus is on solving LP models using spreadsheets. However, we briefly discuss graphical solution procedures for LP models involving only two variables. The graphical solution procedure is useful in that it allows us to provide an intuitive explanation of the procedure used by most software packages to solve LP problems of any size.

Interpretation and Sensitivity Analysis

Sensitivity analysis *allows a manager to answer "what-if" questions regarding a problem's solution.*

Assuming that a formulation is correct and has been successfully implemented and solved using an LP software package, how does a manager use the results? In addition to just providing the solution to the current LP problem, the computer results also allow the manager to evaluate the impact of several different types of what-if questions regarding the problem. We discuss this subject, called *sensitivity analysis*, in Chapter 4.

In this textbook, our emphasis is on formulation (Chapters 2 and 3) and interpretation (Chapter 4), along with detailed descriptions of how spreadsheets can be used to efficiently set up and solve LP models.

Properties of a Linear Programming Model

All LP models have the following properties in common:

First LP property: Problems seek to maximize or minimize an objective.

1. All problems seek to maximize or minimize some quantity, usually profit or cost. We refer to this property as the **objective function** of an LP problem. For example, the objective of a typical manufacturer is to maximize profits. In the case of a trucking or railroad distribution system, the objective might be to minimize shipping costs. In any event, this objective must be stated clearly and defined mathematically. It does not matter whether profits and cost are measured in cents, dollars, euros, or millions of dollars. An *optimal solution* to the problem is the solution that achieves the best value (maximum or minimum, depending on the problem) for the objective function.

Second LP property: Constraints limit the degree to which the objective can be obtained.

2. LP models usually include restrictions, or **constraints**, that limit the degree to which we can pursue our objective. For example, when we are trying to decide how many units to produce of each product in a firm's product line, we are restricted by the available machinery time. Likewise, in selecting food items for a hospital meal, a dietitian must ensure that minimum daily requirements of vitamins, protein, and so on are satisfied. We want, therefore, to maximize or minimize a quantity (the objective) subject to limited resources (the constraints).

   An LP model usually includes a set of constraints known as **nonnegativity constraints**. These constraints ensure that the variables in the model take on only nonnegative values (i.e., ≥ 0). This is logical because negative values of physical quantities are impossible; you simply cannot produce a negative number of chairs or computers.

Third LP property: There must be alternatives available.

3. There must be alternative courses of action from which we can choose. For example, if a company produces three different products, management could use LP to decide how to allocate its limited production resources (of personnel, machinery, and so on) among these products. Should it devote all manufacturing capacity to make only the first product, should it produce equal numbers or amounts of each product, or should it allocate the resources in some other ratio? If there were no alternative to select from, we would not need LP.

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[1] We refer to the best solution as *an* optimal solution rather than as *the* optimal solution because, as we shall see later, the problem could have more than one optimal solution.

Fourth LP property: Mathematical relationships are linear.

4. The objective and constraints in LP problems must be expressed in terms of *linear* equations or inequalities. In linear mathematical relationships, all terms used in the objective function and constraints are of the first degree (i.e., not squared, or to the third or higher power, or appearing more than once). Hence, the equation $2A + 5B = 10$ is a valid linear function, whereas the equation $2A^2 + 5B^3 + AB = 10$ is not linear because the variable A is squared, the variable B is cubed, and the two variables appear as a product in the third term.

An inequality has a ≤ or ≥ sign.

You will see the term **inequality** quite often when we discuss LP problems. By *inequality* we mean that not all LP constraints need be of the form $A + B = C$. This particular relationship, called an equation, implies that the sum of term A and term B exactly equals term C. In most LP problems, we see inequalities of the form $A + B \leq C$ or $A + B \geq C$. The first of these means that A plus B is less than or equal to C. The second means that A plus B is greater than or equal to C. This concept provides a lot of flexibility in defining problem limitations.

Basic Assumptions of a Linear Programming Model

Four technical requirements are certainty, proportionality, additivity, and divisibility.

Technically, there are four additional requirements of an LP problem of which you should be aware:

1. We assume that conditions of *certainty* exist. That is, numbers used in the objective function and constraints are known with certainty and do not change during the period being studied.
2. We also assume that *proportionality* exists in the objective function and constraints. This means that if production of 1 unit of a product uses 3 hours of a particular resource, then making 10 units of that product uses 30 hours of the resource.
3. The third assumption deals with *additivity*, meaning that the total of all activities equals the sum of the individual activities. For example, if an objective is to maximize profit = \$8 per unit of the first product made plus \$3 per unit of the second product made, and if 1 unit of each product is actually produced, the profit contributions of \$8 and \$3 must add up to produce a sum of \$11.
4. We make the *divisibility* assumption that solutions need not necessarily be in whole numbers (integers). That is, they may take any fractional value. If a fraction of a product cannot be produced (e.g., one-third of a submarine), an integer programming problem exists. We discuss integer programming in more detail in Chapter 6.

2.3 Formulating a Linear Programming Problem

Product mix problems use LP to decide how much of each product to make, given a series of resource restrictions.

One of the most common LP applications is the **product mix problem**. In many manufacturing firms, two or more products are usually produced using limited resources, such a personnel, machines, raw materials, and so on. The profit that the firm seeks to maximize is based on the profit contribution per unit of each product. (Profit contribution, you may recall, is the selling price per unit minus the variable cost per unit.[2]) The firm would like to determine how many units of each product it should produce so as to maximize overall profit, given its limited resources.

Problems with only two variables are uncommon in practice.

We begin our discussion of LP formulation with a simple product mix problem that involves only two variables (one for each product, in this case). We recognize that in most real-world situations, there is very little chance we will encounter LP models with just two variables. Such LP models therefore have little *real-world* value. We nevertheless consider it worthwhile to study these models here for two reasons. First, the compact size of these models makes it easier for a beginning student to understand the structure of LP models and the logic behind their formulation. As we will see, the same structure and logic carry forward even to problems of larger size. Second, and more importantly, as we will see in section 2.4, we can represent a two-variable model in a graphical form, which allows us to visualize the interaction between various issues in the problem.

[2] Technically, we maximize total contribution margin, which is the difference between unit selling price and costs that vary in proportion to the quantity of the item produced. Depreciation, fixed general expense, and advertising are excluded from calculations.

Linear Programming Example: Flair Furniture Company

Flair Furniture Company produces inexpensive tables and chairs. The production process for each is similar in that both require a certain number of labor hours in the carpentry department and a certain number of labor hours in the painting department. Each table takes 3 hours of carpentry work and 2 hours of painting work. Each chair requires 4 hours of carpentry and 1 hour of painting. During the current month, 2,400 hours of carpentry time and 1,000 hours of painting time are available. The marketing department wants Flair to make no more than 450 new chairs this month because there is a sizable existing inventory of chairs. However, because the existing inventory of tables is low, the marketing department wants Flair to make at least 100 tables this month. Each table sold results in a profit contribution of \$7, and each chair sold yields a profit contribution of \$5.

Flair Furniture's problem is to determine the best possible combination of tables and chairs to manufacture this month in order to attain the maximum profit. The firm would like this product mix situation formulated (and subsequently solved) as an LP problem.

To provide a structured approach for formulating this problem (and any other LP problem, irrespective of size and complexity), we present a three-step process in the following sections.

Decision Variables

Decision variables *are the unknown entities in a problem. The problem is solved to find values for decision variables.*

Decision variables (or choice variables) represent the unknown entities in a problem—that is, what we are solving for in the problem. For example, in the Flair Furniture problem, there are two unknown entities: the number of tables to be produced this month and the number of chairs to be produced this month. Note that all other unknowns in the problem (e.g., the total carpentry time needed this month) can be expressed as linear functions of the number of tables produced and the number of chairs produced.

Decision variables are expressed in the problems using alphanumeric symbols. When writing the formulation on paper, it is convenient to express the decision variables using simple names that are easy to understand. For example, the number of tables to be produced can be denoted by names such as T, *Tables*, or X_1, and the number of chairs to be produced can be denoted by names such as C, *Chairs*, or X_2.

Throughout this textbook, to the extent possible, we use self-explanatory names to denote the decision variables in our formulations. For example, in Flair Furniture's problem, we use T and C to denote the number of tables and chairs to be produced this month, respectively.

Different decision variables in the same model can be measured in different units.

Although the two decision variables in Flair's model define similar entities (in the sense that they both represent the number of units of a product to make), this need not be the case in all LP (and other) decision models. It is perfectly logical for different decision variables in the same model to define completely different entities and be measured in different units. For example, variable X can denote the amount of labor to use (measured in hours), while variable Y can denote the amount of paint to use (measured in gallons).

The Objective Function

The **objective function** *represents the motivation for solving a problem.*

The objective function states the goal of a problem—that is, why we are trying to solve the problem. An LP model must have a single objective function. In most business-oriented LP models, the objective is to either maximize profit or minimize cost. The goal in this step is to express the profit (or cost) in terms of the decision variables defined earlier. In Flair Furniture's problem, the total profit can be expressed as

$$\begin{aligned}\text{Profit} = {} & (\$7 \text{ profit per table}) \times (\text{number of tables produced}) \\ & + (\$5 \text{ profit per chair}) \times (\text{number of chairs produced})\end{aligned}$$

Using the decision variables T and C defined earlier, the objective function can be written as

$$\text{Maximize } \$7T + \$5C$$

Constraints

Constraints *represent restrictions on the values the decision variables can take.*

Constraints denote conditions that prevent us from selecting any value we please for the decision variables. An LP model can have as many constraints as necessary for a problem scenario. Each constraint is expressed as a mathematical expression and can be independent of the other constraints in the model.

In Flair's problem, we note that there are four restrictions on the solution. The first two have to do with the carpentry and painting times available. The third and fourth constraints deal with marketing-specified production conditions on the numbers of chairs and tables to make, respectively.

With regard to the carpentry and painting times, the constraints must ensure that the amount of the resource (time) required by the production plan is less than or equal to the amount of the resource (time) available. For example, in the case of carpentry, the total time used is

$$(3 \text{ hour per table}) \times (\text{number of tables produced})$$
$$+ (4 \text{ hours per chair}) \times (\text{number of chairs produced})$$

There are 2,400 hours of carpentry time available. Using the decision variables T and C defined earlier, this constraint can be stated as

$$3T + 4C \leq 2{,}400$$

The resource constraints put limits on the carpentry time and painting time needed mathematically.

Likewise, the second constraint specifies that the painting time used is less than or equal to the painting time available. This can be stated as

$$2T + 1C \leq 1{,}000$$

Next, there is the marketing-specified constraint that no more than 450 chairs be produced. This can be expressed as

$$C \leq 450$$

It is common for different constraints to have different signs in an LP model.

Finally, there is the second marketing-specified constraint that at least 100 tables must be produced. Note that unlike the first three constraints, this constraint involves the $\geq$ sign because 100 is a minimum requirement. It is very common in practice for a single LP model to include constraints with different signs (i.e., $\leq$, $\geq$, and $=$). The constraint on the production of tables can be expressed as

$$T \geq 100$$

A key principle of LP is that interactions exist between variables.

All four constraints represent restrictions on the numbers that we can make of the two products and, of course, affect the total profit. For example, Flair cannot make 900 tables because the carpentry and painting constraints are both violated if $T = 900$. Likewise, it cannot make 500 tables and 100 chairs, because that would require more than 1,000 hours of painting time. Hence, we note one more important aspect of LP models: Certain interactions exist between variables. The more units of one product that a firm produces, the fewer it can make of other products. We show how this concept of interaction affects the solution to the model as we tackle the graphical solution approach in the next section.

Nonnegativity Constraints and Integer Values

Nonnegativity constraints specify that decision variables cannot have negative values.

Before we consider the graphical solution procedure, there are two other issues we need to address. First, because Flair cannot produce negative quantities of tables or chairs, the nonnegativity constraints must be specified. Mathematically, these can be stated as

$$T \geq 0 \qquad (\text{number of tables produced} \geq 0)$$
$$C \geq 0 \qquad (\text{number of chairs produced} \geq 0)$$

In LP models, we do not specify that decision variables should only have integer values.

Second, it is possible that the optimal solution to the LP model will result in fractional values for T and C. Because the production plan in Flair's problem refers to a month's schedule, we can view fractional values as work-in-process inventory that is carried over to the next month. However, in some problems, we may require the values for decision variables to be whole numbers (integers) in order for the solution to make practical sense. A model in which some or all of the decision variables are restricted only to integer values is called an **integer programming (*IP*)** model. We will study IP models in detail in Chapter 6. In general, as we will see in Chapter 6, it is considerably more difficult to solve an IP problem than an LP problem. Further, LP model solutions allow detailed sensitivity analysis (the topic of Chapter 4) to be undertaken, whereas IP model solutions do not. For these reasons, we do not specify the integer requirement in LP models, and we permit fractional values in the solution. Fractional values can then be rounded off appropriately, if necessary.

Guidelines to Developing a Correct LP Model

We have now developed our first LP model. Before we proceed further, let us address a question that many students have, especially at the early stages of their experience with LP formulation: "How do I know my LP model is right?" There is, unfortunately, no simple magical answer for this question. Instead, we offer the following guidelines that students can use to judge on their own whether their model does what it is intended to do. Note that these guidelines do not guarantee that your model is correct. Formulation is still an art that you master only through repeated application to several diverse problems. (We will practice this over the next few chapters.) However, by following these guidelines, you can hopefully avoid the common errors that many beginners commit:

Here are a few guidelines to developing a correct LP model.

- Recognizing and defining the decision variables is perhaps the most critical step in LP formulation. In this endeavor, one approach we have often found useful and effective is to assume that you have to communicate your result to someone else. When you tell that person "The answer is to do ——————," what exactly do you need to know to fill in the blank? Those entities are usually the decision variables.
- Remember that it is perfectly logical for different decision variables in a single LP model to be measured in different units. That is, all decision variables in an LP model need not denote similar entities.
- All expressions in the model (the objective function and each constraint) must use *only* the decision variables that have been defined for the model. For example, in the Flair Furniture problem, the decision variables are T and C. Notice that all expressions involve only T and C. It is, of course, permissible for a decision variable to not be part of a specific expression. For example, the variable T is not part of the constraint $C \leq 450$.
- At any stage of the formulation, if you find yourself unable to write a specific expression (the objective function or a constraint) using the defined decision variables, it is a pretty good indication that you either need more decision variables or you've defined your decision variables incorrectly.
- All terms within the same expression must refer to the same entity. Consider, for example, the expression for the carpentry constraint $3T + 4C \leq 2{,}400$. Notice that each term (i.e., $4T$, $3C$, and 2,400) measures an amount of carpentry time. Likewise, in the objective function, each term (i.e., \$7T and \$5C) in the expression measures profit.
- All terms within the same expression must be measured in the same units. That is, if the first term in an expression is in hours, all other terms in that expression must also be in hours. For example, in the carpentry constraint $3T + 4C \leq 2{,}400$, the $4T$, $3C$, and 2,400 are each measured in hours.
- Address each constraint separately. That is, there is no single "mega" expression that will take care of all constraints in the model at one time. Each constraint is a separate issue,

IN ACTION: Linear Programing Helps General Electric Units to Optimize Portfolios

General Electric Asset Management Incorporated (GEAM) manages investment portfolios worth billions of dollars on behalf of various General Electric (GE) units and other clients worldwide, including Genworth Financial and GE Insurance (GEI). GEAM, a wholly owned subsidiary of GE, invests portfolios of assets primarily in corporate and government bonds, taking into account risk and regulatory constraints. The objective is to identify the portfolios' risk/return trade-offs by maximizing the return or minimizing the risk. While risk is widely represented by variance or volatility, it is usually a nonlinear measure and portfolio managers typically use linear risk sensitivities for computational tractability.

To address this problem, a multidisciplinary team from GE Global Research Center worked with GEAM, Genworth, and GEI to develop a sequential linear-programming algorithm that handles the risk nonlinearity iteratively but efficiently. The team determined that the optimal solution for the portfolio management problem would result in improved financial performance and better understanding of the risk/return trade-off. GE initially used the algorithm on a limited basis to optimize portfolios valued at over \$30 billion. It is now in broader use at GEAM, GEI, and Genworth. It is estimated that for every \$100 billion of assets, the present value of potential benefits is around \$75 million over five years.

Source: Based on K. C. Chalermkraivuth et al. "GE Asset Management, Genworth Financial, and GE Insurance Use a Sequential-Linear-Programming Algorithm to Optimize Portfolios," *Interfaces* 35, 5 (September-October 2005): 370–380.

and you must write a separate expression for each one. While writing one constraint (e.g., carpentry time), do not worry about other constraints (e.g., painting time).

- Try "translating" the mathematical expression back to words. After all, writing a constraint is just a matter of taking a problem scenario that is in words (e.g., "the amount of the carpentry time required by the production plan should be less than or equal to the carpentry time available") and translating it to a simple linear mathematical expression (e.g., $3T + 4C \leq 2{,}400$). To make sure the translation has been done correctly, do the reverse process. That is, try explaining in words (to yourself) what the expression you have just written is saying. While doing so, make sure you remember the previous guidelines about all terms in an expression dealing with the same issue and being measured in the same units. If your "reverse translation" yields exactly the situation that you were trying to express in mathematical form, chances are your expression is correct.

2.4 Graphical Solution of a Linear Programming Problem with Two Variables

The graphical method works only when there are two decision variables, but it provides valuable insight into how larger problems are solved.

As noted earlier, there is little chance of encountering LP models with just two variables in real-world situations. However, a major advantage of two-variable LP models (such as Flair Furniture's problem) is that they can be graphically illustrated using a two-dimensional graph. This graph can then be used to identify the optimal solution to the model. Although this graphical solution procedure has limited value in real-world situations, it is invaluable in two respects. First, it provides insights into the properties of solutions to *all* LP models, regardless of their size. Second, even though we use a computerized spreadsheet based procedure to solve LP models in this textbook, the graphical procedure allows us to provide an intuitive explanation of how this more complex solution procedure works for LP models of any size. For these reasons, we first discuss the solution of Flair's problem using a graphical approach.

Graphical Representation of Constraints

Here is a complete mathematical statement of the Flair LP problem.

The complete LP model for Flair's problem can be restated as follows:

$$\text{Maximize profit} = \$7T + \$5C$$

subject to the constraints

$$\begin{aligned} 3T + 4C &\leq 2{,}400 && \text{(carpentry time)} \\ 2T + 1C &\leq 1{,}000 && \text{(painting time)} \\ C &\leq 450 && \text{(maximum chairs allowed)} \\ T &\geq 100 && \text{(minimum tables required)} \\ T, C &\geq 0 && \text{(nonnegativity)} \end{aligned}$$

Nonnegativity constraints mean we are always in the graphical area where $T \geq 0$ *and* $C \geq 0$.

To find an optimal solution to this LP problem, we must first identify a set, or region, of feasible solutions. The first step in doing so is to plot each of the problem's constraints on a graph. We can plot either decision variable on the horizontal (X) axis of the graph, and the other variable on the vertical (Y) axis. In Flair's case, let us plot T (tables) on the X-axis and C (chairs) on the Y-axis. The nonnegativity constraints imply that we are working only in the first (or positive) quadrant of a graph.

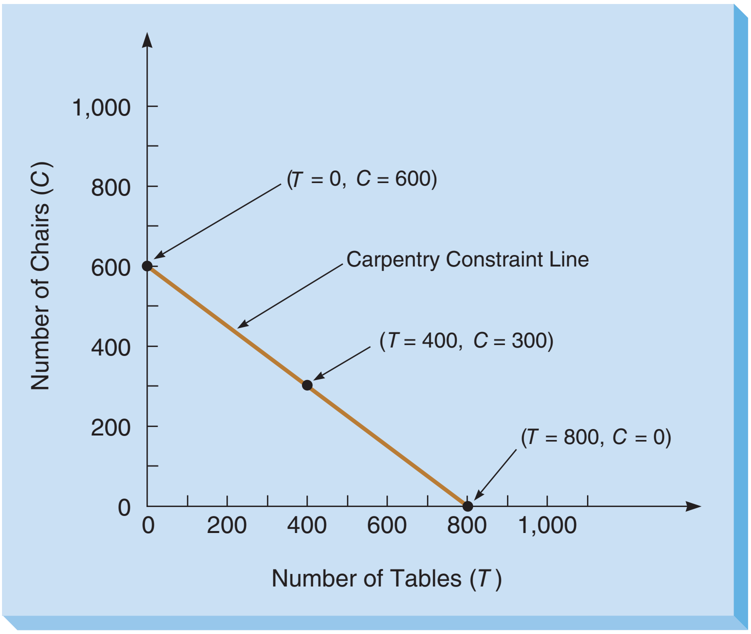
CARPENTRY TIME CONSTRAINT To represent the carpentry constraint graphically, we first convert the expression into a linear equation (i.e., $3T + 4C = 2{,}400$) by replacing the inequality sign ($\leq$) with an equality sign ($=$).

Plotting the first constraint involves finding points at which the line intersects the T*-axis and* C*-axis.*

As you may recall from elementary algebra, the solution of a linear equation with two variables represents a straight line. The easiest way to plot the line is to find any two points that satisfy the equation and then draw a straight line through them. The two easiest points to find are generally the points at which the line intersects the horizontal (T) and vertical (C) axes.

If Flair produces no tables (i.e., $T = 0$), then $3(0) + 4C = 2{,}400$, or $C = 600$. That is, the line representing the carpentry time equation crosses the vertical axis at $C = 600$. This indicates that if the entire carpentry time available is used to make only chairs, Flair could make 600 chairs this month.

FIGURE 2.1
Graph of the Nonnegativity Constraint and the Carpentry Constraint Equation

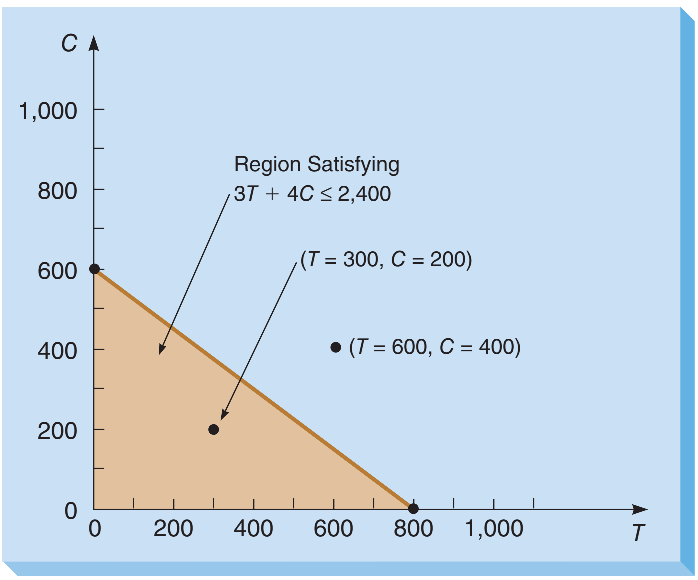


To find the point at which the line $3T + 4C = 2{,}400$ crosses the horizontal axis, let us assume that Flair uses all the carpentry time available to make only tables. That is, $C = 0$. Then $3T + 4(0) = 2{,}400$, or $T = 800$.

The nonnegativity constraints and the carpentry constraint line are illustrated in Figure 2.1. The line running from point $(T = 0, C = 600)$ to point $(T = 800, C = 0)$ represents the carpentry time equation $3T + 4C = 2{,}400$. We know that any combination of tables and chairs represented by points on this line (e.g., $T = 400$, $C = 300$) will use up all 2,400 hours of carpentry time.[3]

Recall, however, that the actual carpentry constraint is the inequality $3T + 4C \leq 2{,}400$. How do we identify all the points on the graph that satisfy this inequality? To do so, we check any possible point in the graph. For example, let us check $(T = 300, C = 200)$. If we substitute these values in the carpentry constraint, the result is $3 \times 300 + 4 \times 200 = 1{,}700$. Because 1,700 is less than 2,400, the point $(T = 300, C = 200)$ satisfies the inequality. Further, note in Figure 2.2 that this point is below the constraint line.

FIGURE 2.2
Region That Satisfies the Carpentry Constraint



[3] Thus, we have plotted the carpentry constraint equation in its most binding position (i.e., using all of the resource).

There is a whole region of points that satisfies the first inequality constraint.

In contrast, let's say the point we select is $(T = 600, C = 400)$. If we substitute these values in the carpentry constraint, the result is $3 \times 600 + 4 \times 400 = 3{,}400$. Because 3,400 exceeds 2,400, this point violates the constraint and is, therefore, an unacceptable production level. Further, note in Figure 2.2 that this point is above the constraint line. As a matter of fact, any point above the constraint line violates that restriction (test this yourself with a few other points), just as any point below the line does not violate the constraint. In Figure 2.2, the shaded region represents all points that satisfy the carpentry constraint inequality $3T + 4C \le 2{,}400$.

PAINTING TIME CONSTRAINT Now that we have identified the points that satisfy the carpentry constraint, we recognize that the final solution must also satisfy all other constraints in the problem. Therefore, let us now add to this graph the solution that corresponds to the painting constraint.

Recall that we expressed the painting constraint as $2T + 1C \le 1{,}000$. As we did with the carpentry constraint, we start by changing the inequality to an equation and identifying two points on the line specified by the equation $2T + 1C = 1{,}000$. When $T = 0$, then $2(0) + 1C = 1{,}000$, or $C = 1{,}000$. Likewise, when $C = 0$, then $2T + 1(0) = 1{,}000$, or $T = 500$.

The line from the point $(T = 0, C = 1{,}000)$ to the point $(T = 500, C = 0)$ in Figure 2.3 represents all combinations of tables and chairs that use exactly 1,000 hours of painting time. As with the carpentry constraint, all points on or below this line satisfy the original inequality $2T + 1C \le 1{,}000$.

In Figure 2.3, some points, such as $(T = 300, C = 200)$, are below the lines for both the carpentry equation and the painting equation. That is, we have enough carpentry and painting time available to manufacture 300 tables and 200 chairs this month. In contrast, there are points, such as $(T = 500, C = 200)$ and $(T = 100, C = 700)$, that satisfy one of the two constraints but violate the other. (See if you can verify this statement mathematically.) Because we need the solution to satisfy both the carpentry and painting constraints, we will consider only those points that satisfy both constraints simultaneously. The region that contains all such points is shaded in Figure 2.3.

PRODUCTION CONSTRAINT FOR CHAIRS We have to make sure the final solution requires us to make no more than 450 chairs $(C \le 450)$. As before, we first convert this inequality to an equation $(C = 450)$. This is relatively easy to draw because it is just a horizontal line that intersects the vertical (C) axis at 450. This line is shown in Figure 2.4, and all points below this line satisfy the original inequality $(C \le 450)$.

PRODUCTION CONSTRAINT FOR TABLES Finally, we have to ensure that the final solution makes at least 100 tables $(T \ge 100)$. In this case, the equation $(T = 100)$ is just a vertical line

FIGURE 2.3
Region That Satisfies the Carpentry and Painting Constraints

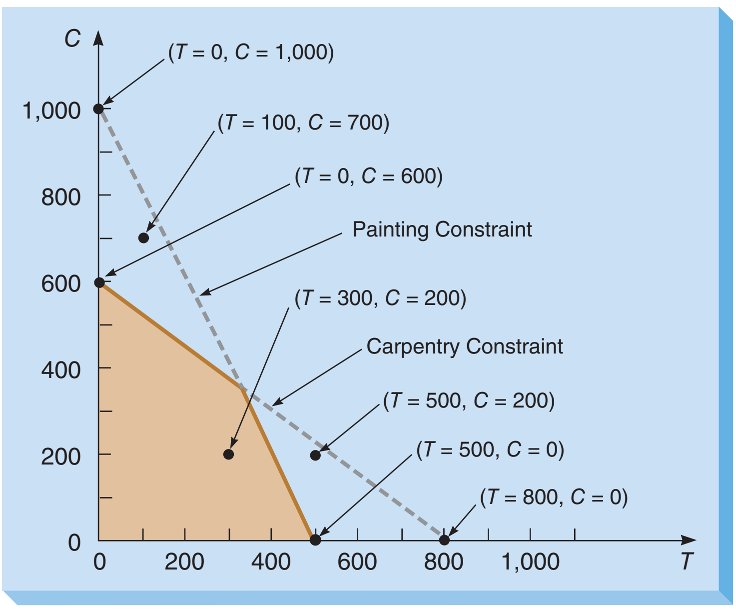
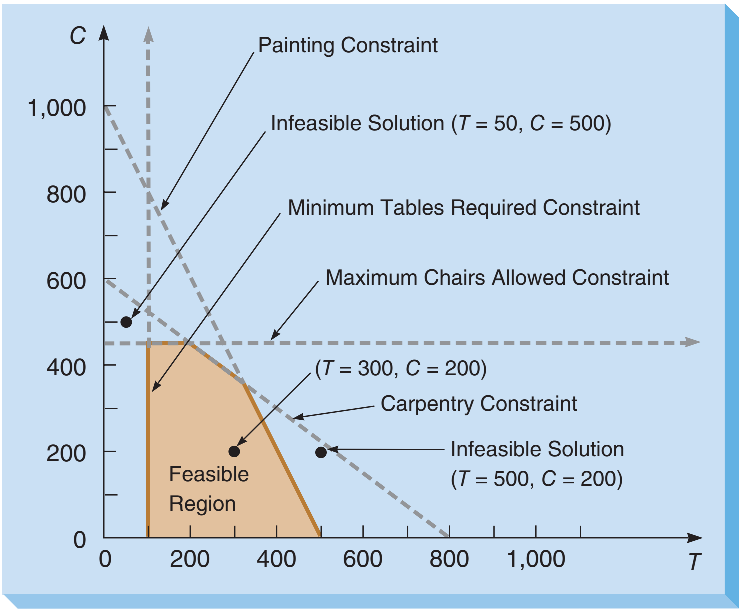


FIGURE 2.4
Feasible Solution Region for the Flair Furniture Company Problem



that intersects the horizontal (T) axis at 100. This line is also shown in Figure 2.4. However, because this constraint has the $\geq$ sign, it should be easy to verify that all points to the *right* of this line satisfy the original inequality ($T \geq 100$).

Feasible Region

In all problems, we are interested in satisfying all constraints at the same time.

The **feasible region** *is the overlapping area of all constraints.*

The **feasible region** of an LP problem consists of those points that simultaneously satisfy all constraints in the problem; that is, it is the region where all the problem's constraints overlap.

Consider a point such as ($T = 300, C = 200$) in Figure 2.4. This point satisfies all four constraints, as well as the nonnegativity constraints. This point, therefore, represents a **feasible solution** to Flair's problem. In contrast, points such as ($T = 500, C = 200$) and ($T = 50, C = 500$) each violate one or more constraints. They are, therefore, not feasible solutions. The shaded area in Figure 2.4 represents the feasible region for Flair Furniture's problem. Any point outside the shaded area represents an **infeasible solution** (or production plan).

Identifying an Optimal Solution by Using Level Lines

When the feasible region has been identified, we can proceed to find the optimal solution to the problem. In Flair's case, the *optimal solution* is the point in the feasible region that produces the highest profit. But there are many, many possible solution points in the feasible region. How do we go about selecting the optimal one, the one that yields the highest profit? We do this by essentially using the objective function as a "pointer" to guide us toward an optimal point in the feasible region.

We use the objective function to point us toward the optimal solution.

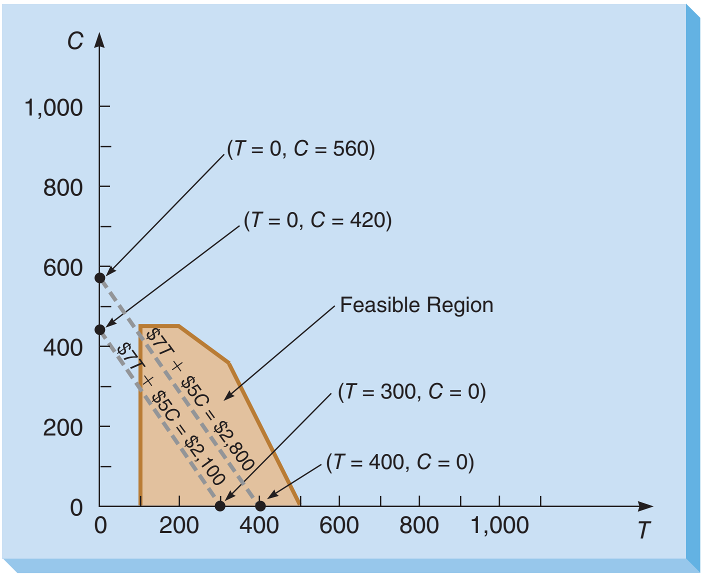
DRAWING LEVEL LINES In the **level, or iso, lines** method, we begin by plotting the line that represents the objective function (i.e., $\$7T + \$5C$) on the graph, just as we plotted the various constraints.[4] However, note that we do not know what $\$7T + \$5C$ equals in this function. In fact, that's what we are trying to find out. Without knowing this value, how do we plot this equation?

To get around this problem, let us first write the objective function as $\$7T + \$5C = Z$. We then start the procedure by selecting *any* arbitrary value for Z. In selecting this value for Z, the only recommended guideline is to select a value that makes the resulting equation easy to plot on the graph. For example, for Flair's problem, we can choose a profit of \$2,100. We can then write the objective function as $\$7T + \$5C = \$2{,}100$.

Clearly, this expression is the equation of a line that represents all combinations of (T, C) that would yield a total profit of \$2,100. That is, it is a *level line* corresponding to a profit of \$2,100.

[4] Iso means "equal" or "similar." Thus, an isoprofit line represents a line with all profits the same, in this case \$2,100.

FIGURE 2.5
Level Profit Lines for $Z = \$2{,}100$ and $Z = \$2{,}800$



To plot this line, we proceed exactly as we do to plot a constraint line. If we let $T = 0$, then $\$7(0) + \$5C = \$2{,}100$, or $C = 420$. Likewise, if we let $C = 0$, then $\$7T + \$5(0) = \$2{,}100$, or $T = 300$.

The objective function line corresponding to $Z = \$2{,}100$ is illustrated in Figure 2.5 as the line between $(T = 0, C = 420)$ and $(T = 300, C = 0)$. Observe that if any points on this line lie in the feasible region identified earlier for Flair's problem, those points represent *feasible production plans* that will yield a profit of \$2,100.

What if we had selected a different Z value, such as \$2,800, instead of \$2,100? In that case, the objective function line corresponding to $Z = \$2{,}800$ would be between the points $(T = 0, C = 560)$ and $(T = 400, C = 0)$, also shown in Figure 2.5. Further, because there are points on this line that lie within the feasible region for Flair's problem, it is possible for Flair to find a production plan that will yield a profit of \$2,800 (obviously, better than \$2,100).

Observe in Figure 2.5 that the level lines for $Z = \$2{,}100$ and $Z = \$2{,}800$ are parallel to each other. This is a very important point. It implies that regardless of which value of Z we select, the objective function line that we draw will be parallel to the two level lines shown in Figure 2.5. The exact location of the parallel line on the graph will, of course, depend on the value of Z selected.

We draw a series of parallel level lines until we find the one that corresponds to the optimal solution.

We know now that Flair can obtain a profit of \$2,800. However, is \$2,800 the highest profit that Flair can get? From the preceding discussion, we note that as the value we select for Z gets larger (which is desirable in Flair's problem because we want to maximize profit), the objective function line moves in a *parallel* fashion away from the origin. Therefore, we can "draw" a series of parallel level lines (by carefully moving a ruler in a plane parallel to the $Z = \$2{,}800$ line). However, as we visualize these parallel lines, we need to ensure that at least one point on each level line lies within the feasible region. The level line that corresponds to the highest profit but still touches some point of the feasible region pinpoints an optimal solution.

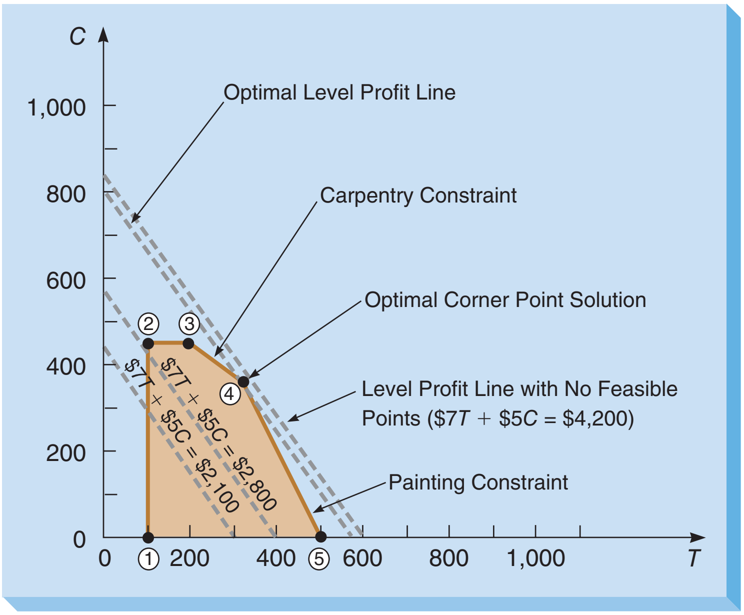
From Figure 2.6, we can see that the level profit line that corresponds to the highest achievable profit value will be tangential to the shaded feasible region at the point denoted by ④. Any level line corresponding to a profit value higher than that of this line will have no points in the feasible region. For example, note that a level line corresponding to a profit value of \$4,200 is entirely outside the feasible region (see Figure 2.6). This implies that a profit of \$4,200 is not possible for Flair to achieve.

Observe that point ④ defines the intersection of the carpentry and painting constraint equations. Such points, where two or more constraints intersect, are called **corner points, or extreme points**. In Figure 2.6, note that the other corner points in Flair's problem are points ①, ②, ③, and ⑤.

An optimal solution to an LP model must lie at one of the corner points in the feasible region.

CORNER POINT PROPERTY The preceding discussion reveals an important property of LP problems, known as the *corner point property*. This property states that an optimal solution to an LP problem will always occur at a corner point of the feasible region. In Flair's problem,

FIGURE 2.6
Optimal Corner Point Solution to the Flair Furniture Company Problem



this means that the optimal solution has to be one of the five corner points (i.e., ①, ②, ③, ④, or ⑤). For the specific objective function considered here (Maximize $7T + 5C$), corner point ④ turns out to be optimal. For a different objective function, one of the other corner points could be optimal.

CALCULATING THE SOLUTION AT AN OPTIMAL CORNER POINT Now that we have identified point ④ in Figure 2.6 as an optimal corner point, how do we find the values of T and C, and the profit at that point? Of course, if a graph is perfectly drawn, you can always find point ④ by carefully examining the intersection's coordinates. Otherwise, the algebraic procedure shown here provides more precision.

Solving for the coordinates of a corner point requires the use of simultaneous equations, an algebraic technique.

To find the coordinates of point ④ accurately, we have to solve for the intersection of the two constraint equations intersecting at that point. Recall from your last course in algebra that you can apply the **simultaneous equations method** to the two constraint equations:

$$3T + 4C = 2{,}400 \qquad \text{(carpentry time equation)}$$
$$2T + 1C = 1{,}000 \qquad \text{(painting time equation)}$$

To solve these equations simultaneously, we need to eliminate one of the variables and solve for the other. One way to do this would be to first multiply the first equation by 2 and the second equation by 3. If we then subtract the modified second equation from the modified first equation, we get

$$\begin{array}{r} 6T + 8C = 4{,}800 \\ \underline{-(6T + 3C = 3{,}000)} \\ 5C = 1{,}800 \end{array} \qquad \text{implies } C = 360$$

We can now substitute 360 for C in either of the original equations and solve for T. For example, if $C = 360$ in the first equation, then $3T + (4)(360) = 2{,}400$, or $T = 320$. That is, point ④ has the coordinates $(T = 320, C = 360)$. Hence, in order to maximize profit, Flair Furniture should produce 320 tables and 360 chairs. To complete the analysis, we can compute the optimal profit as $\$7 \times 320 + \$5 \times 360 = \$4{,}040$.

Identifying an Optimal Solution by Using All Corner Points

Because an optimal solution to any LP problem always occurs at a corner point of the feasible region, we can identify an optimal solution by evaluating the objective function value at every corner point in the problem. While this approach, called the **corner point method**, eliminates the need for graphing and using level objective function lines, it is somewhat tedious because we end up unnecessarily identifying the coordinates of many corner points. Nevertheless, some people prefer this approach because it is conceptually much simpler than the level lines approach.

To verify the applicability of this approach to Flair's problem, we note from Figure 2.6 that the feasible region has five corner points: ①, ②, ③, ④, and ⑤. Using the procedure discussed earlier for corner point ④, we find the coordinates of each of the other four corner points and compute their profit levels. They are as follows:

Point ①	$(T = 100, C = 0)$	Profit $= \$7 \times 100 + \$5 \times 0 = \$700$
Point ②	$(T = 100, C = 450)$	Profit $= \$7 \times 100 + \$5 \times 450 = \$2{,}950$
Point ③	$(T = 200, C = 450)$	Profit $= \$7 \times 200 + \$5 \times 450 = \$3{,}650$
Point ④	$(T = 320, C = 360)$	Profit $= \$7 \times 320 + \$5 \times 360 = \$4{,}040$
Point ⑤	$(T = 500, C = 0)$	Profit $= \$7 \times 500 + \$5 \times 0 = \$3{,}500$

Note that corner point ④ produces the highest profit of any corner point and is therefore the optimal solution. As expected, this is the same solution we obtained using the level lines method.

Comments on Flair Furniture's Optimal Solution

The result for Flair's problem reveals an interesting feature. Even though chairs provide a smaller profit contribution ($5 per unit) than tables ($7 per unit), the optimal solution requires us to make more units of chairs (360) than tables (320). This is a common occurrence in such problems. We cannot assume that we will always produce greater quantities of products with higher profit contributions. We need to recognize that products with higher profit contributions may also consume larger amounts of resources, some of which may be scarce. Hence, even though we may get smaller profit contributions per unit from other products, we may more than compensate for this by being able to make more units of these products.

Notice, however, what happens if the profit contribution for chairs is only $3 per unit instead of $5 per unit. The objective is to now maximize $\$7T + \$3C$ instead of $\$7T + \$5C$. Although Figure 2.6 does not show the profit line corresponding to $\$7T + \$3C$, you should be able to use a straight edge to represent this revised profit line in this figure and verify that the optimal solution will now correspond to corner point ⑤. That is, the optimal solution is to make 500 tables and 0 chairs, for a total profit of $3,500. Clearly, in this case, the profit contributions of tables and chairs are such that we should devote all our resources to making only the higher profit contribution product, tables.

The key point to note here is that in either situation (i.e., when the profit contribution of chairs is $5 per unit and when it is $3 per unit), there is no easy way to predict *a priori* what the optimal solution is going to be with regard to the numbers of tables and chairs to make. We are able to determine these values only after we have formulated the LP model and solved it in each case. This clearly illustrates the power and usefulness of such types of decision models. As you can well imagine, this issue is going to become even more prominent when we deal in subsequent chapters with models that have more than two decision variables.

Extension to Flair Furniture's LP Model

As noted in Chapter 1, the decision modeling process is iterative in most real-world situations. That is, the model may need to be regularly revised to reflect new information. With this in mind, let us consider the following revision to the Flair Furniture model before we move on to the next example.

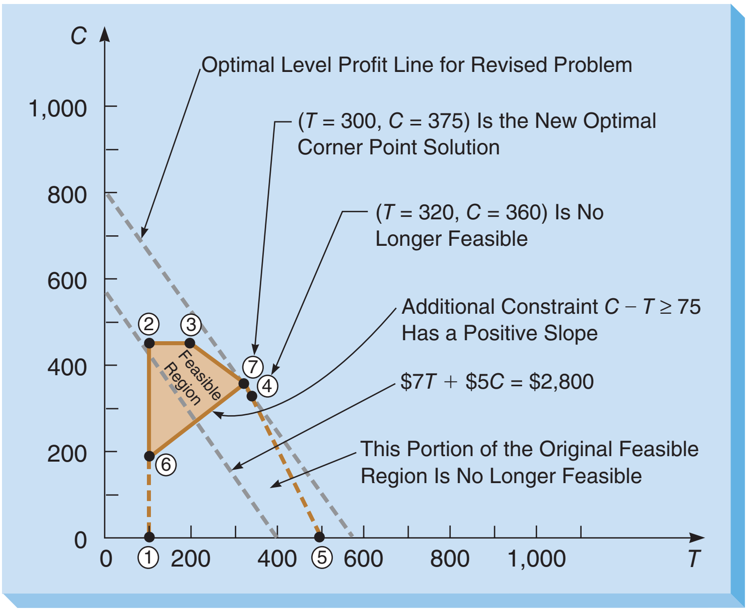
Suppose the marketing department has now informed Flair that all customers purchasing tables usually purchase at least two chairs at the same time. While the existing inventory of chairs may be enough to satisfy a large portion of this demand, the marketing department would like the production plan to ensure that at least 75 more chairs are made this month than tables. Does this new condition affect the optimal solution? If so, how?

Using the decision variables T and C we have defined for Flair's model, we can express this new condition as

$$C \geq T + 75$$

Notice that unlike all our previous conditions, this expression has decision variables on both sides of the inequality. This is a perfectly logical thing to have in an expression, and it does not

FIGURE 2.7
Optimal Corner Point Solution to the Extended Flair Furniture Company Problem



affect the validity of the model in any way. We can, of course, manipulate this expression algebraically if we wish and rewrite it as

$$C - T \geq 75$$

The revised graphical representation of Flair's model due to the addition of this new constraint is shown in Figure 2.7. The primary issue that is noticeably different in drawing the new constraint when compared to the carpentry and painting constraints is that it has a positive slope. All points above this line satisfy the inequality $(C - T \geq 75)$.

Notice the dramatic change in the shape and size of the feasible region just because of this single new constraint. This is a common feature in LP models, and it illustrates how each constraint in a model is important because it can affect the feasible region (and hence, the optimal solution) in a significant manner. In Flair's model, the original optimal corner point 4 $(T = 320, C = 360)$ is no longer even feasible in the revised problem. In fact, of the original corner points, only points 2 and 3 are still feasible. Two new corner points, 6 and 7, now exist.

To determine which of these four corner points (2, 3, 6, and 7 is the new optimal solution, we use a level profit line as before. Figure 2.7 shows the level line for a profit value of $2,800. Based on this line, it appears that corner point 7 is the new optimal solution. The values at this corner point can be determined to be $T = 300$ and $C = 375$, for a profit of $3,975. (See if you

IN ACTION Using Linear Programming to Improve Capacity Management at Indian Railways

Indian Railways (IR) operates more than 1,600 long distance trains and carries more than 7 million passengers daily. Reserved tickets are booked through IR's passenger reservation system, which reserves a specific seat in a specific class on a specific train per booking. A major problem is deciding how many seats to allocate in a given class of a train to multiple travel segments, including segments on which en route passengers (i.e., those who are not traveling from the train's origin to its destination) travel. A train's capacity must therefore be distributed among various intermediate stations by allocating specific quotas to ensure that the twin objectives of maximizing the number of confirmed seats and increasing the seat utilization are met.

IR personnel used a linear programming model to determine the optimal capacity allocation on multiple travel segments. The model, which uses a simple, effective capacity management tool, has helped IR reduce its overall seat requirements and has increased the availability of confirmed seats for the various en route passenger demands on several trains. A spokesperson for IR notes that "The model and software developed have been used in over 50 long-distance trains originating on Western Railway with considerable success."

Source: Based on R. Gopalakrishnan and N. Rangaraj. "Capacity Management on Long-Distance Passenger Trains of Indian Railways," *Interfaces* 40, 4 (July-August 2010): 291–302.

can verify these yourself.) Note that the profit has decreased due to the addition of this new constraint. This is logical because each new constraint could make the feasible region a bit more restrictive. In fact, the best we can hope for when we add a new constraint is that our current optimal solution continues to remain feasible (and hence, optimal).

2.5 A Minimization Linear Programming Problem

Many LP problems involve minimizing an objective such as cost instead of maximizing a profit function. A restaurant, for example, may wish to develop a work schedule to meet staffing needs while minimizing the total number of employees. A manufacturer may seek to distribute its products from several factories to its many regional warehouses in such a way as to minimize total shipping costs. A hospital may want to provide its patients with a daily meal plan that meets certain nutritional standards while minimizing food purchase costs.

Minimization LP problems typically deal with trying to reduce costs.

To introduce the concept of minimization problems, we first discuss a problem that involves only two decision variables. As before, even though such problems may have limited applicability in real-world situations, a primary reason to study them is that they can be represented and solved graphically. This will make it easier for us to understand the structure and behavior of such problems when we consider larger minimization problems in subsequent chapters.

Let's take a look at a common LP problem, referred to as the *diet problem*. This situation is similar to the one that the hospital faces in feeding its patients at the least cost.

Holiday Meal Turkey Ranch

The Holiday Meal Turkey Ranch is planning to use two different brands of turkey feed—brand A and brand B—to provide a good diet for its turkeys. Each feed contains different quantities (in units) of the three nutrients (protein, vitamin, and iron) essential for fattening turkeys. Table 2.1 summarizes this information and also shows the minimum unit of each nutrient required per month by a turkey. Brand A feed costs \$0.10 per pound, and brand B feed costs \$0.15 per pound. The owner of the ranch would like to use LP to determine the quantity of each feed to use in a turkey's diet in order to meet the minimum monthly requirements for each nutrient at the lowest cost.

Here is a complete mathematical formulation of the Holiday Meal LP problem.

If we let A denote the number of pounds of brand A feed to use per turkey each month and B denote the number of pounds of brand B feed to use per turkey each month, we can proceed to formulate this LP problem as follows:

$$\text{Minimize cost} = \$0.10A + \$0.15B$$

subject to the constraints

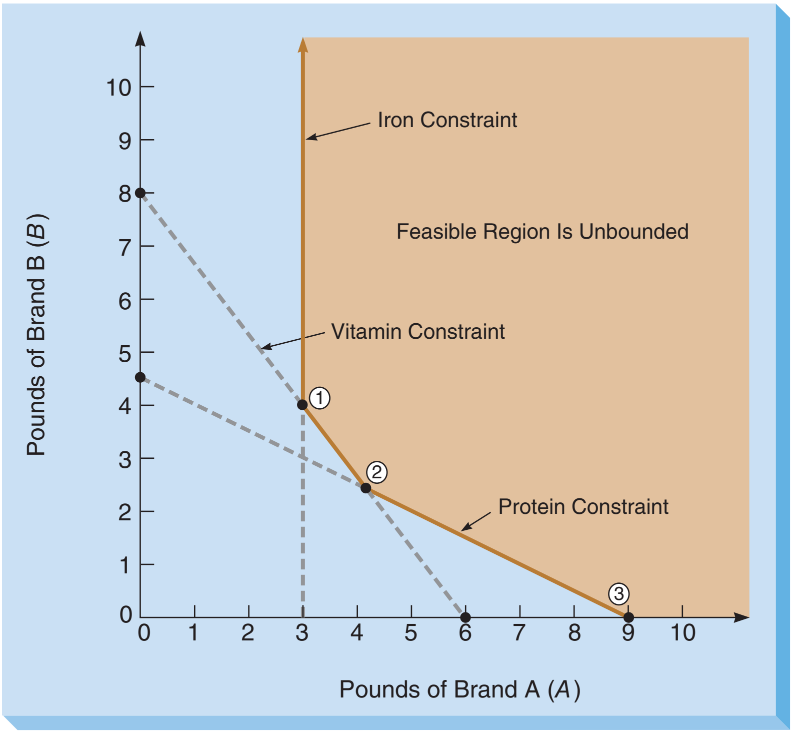
$$\begin{aligned}
5A + 10B &\geq 45 && \text{(protein required)}\\
4A + 3B &\geq 24 && \text{(vitamin required)}\\
0.5A &\geq 1.5 && \text{(iron required)}\\
A, B &\geq 0 && \text{(nonnegativity)}
\end{aligned}$$

Before solving this problem, note two features that affect its solution. First, as the problem is formulated presently, we will be solving for the optimal amounts of brands A and B to use per month *per turkey*. If the ranch houses 5,000 turkeys in a given month, we can simply multiply the A and B quantities by 5,000 to decide how much feed to use overall. Second, we

TABLE 2.1
Data for Holiday Meal Turkey Ranch

NUTRIENT	NUTRIENTS PER POUND OF FEED: BRAND A FEED	NUTRIENTS PER POUND OF FEED: BRAND B FEED	MINIMUM REQUIRED PER TURKEY PER MONTH
Protein (units)	5	10	45.0
Vitamin (units)	4	3	24.0
Iron (units)	0.5	0	1.5
Cost per pound	\$0.10	\$0.15	

FIGURE 2.8
Feasible Region for the Holiday Meal Turkey Ranch Problem



are now dealing with a series of greater than or equal to constraints. These cause the feasible solution area to be above the constraint lines, a common situation when handling minimization LP problems.

Graphical Solution of the Holiday Meal Turkey Ranch Problem

We plot the three constraints to develop a feasible solution region for the minimization problem.

We first construct the feasible solution region. To do so, we plot each of the three constraint equations as shown in Figure 2.8. In plotting constraint such as $0.5A \geq 1.5$, if you find it more convenient to do so, you can multiply both sides by 2 and rewrite the inequality as $A \geq 3$. Clearly, this does not change the position of the constraint line in any way.

Note that the feasible region of minimization problems is often unbounded (i.e., open outward).

The feasible region for Holiday Meal's problem is shown by the shaded space in Figure 2.8. Notice that the feasible region has explicit boundaries inward (i.e., on the left side and the bottom) but is unbounded outward (i.e., on the right side and on top). Minimization problems often exhibit this feature. However, this causes no difficulty in solving them as long as an optimal corner point solution exists on the bounded side. (Recall that an optimal solution will lie at one of the corner points, just as it did in a maximization problem.)

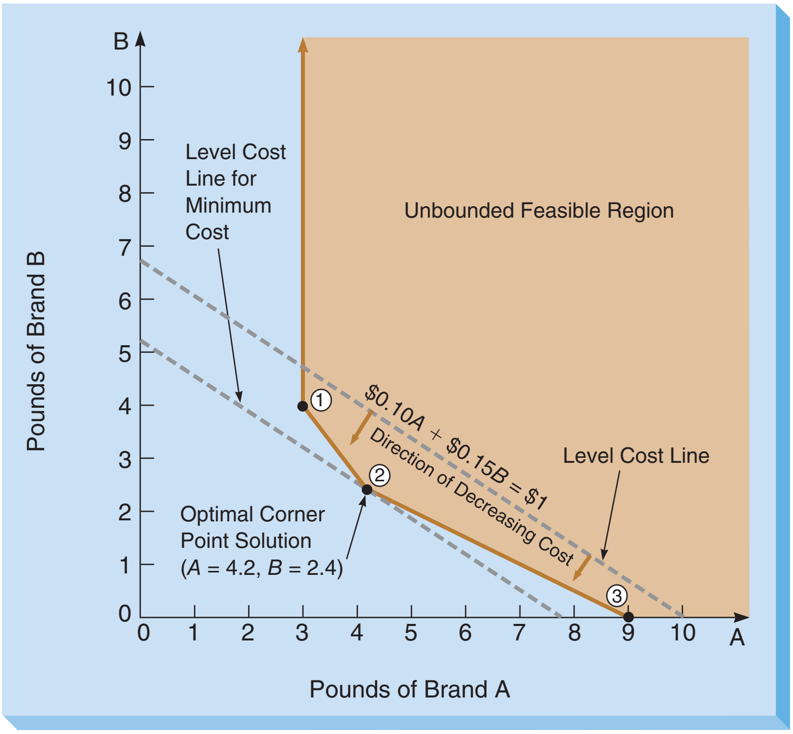
As in maximization problems, we can use the level lines method in a minimization problem to identify an optimal solution.

In Figure 2.8, the identifiable corner points for Holiday Meal's problem are denoted by points ①, ②, and ③. Which, if any, of these corner points is an optimal solution? To answer this, we write the objective function as $\$0.10A + 0.15B = Z$ and plot this equation for *any* arbitrary value of Z. For example, we start in Figure 2.9 by drawing the level cost line corresponding to $Z = \$1.00$. Obviously, there are many points in the feasible region that would yield a lower total cost. As with the parallel level lines we used to solve the Flair Furniture maximization problem, we can draw a series of parallel level cost lines to identify Holiday Meal's optimal solution. The lowest level cost line to touch the feasible region pinpoints an optimal corner point.

Because Holiday Meal's problem involves minimization of the objective function, we need to move our level cost line toward the lower left in a plane parallel to the $1.00 level line. Note that we are moving toward the bounded side of the feasible region and that there are identifiable corner points on this side. Hence, even though the feasible region is unbounded, it is still possible to identify an optimal solution for this problem.

As shown in Figure 2.9, the last feasible point touched by a level cost line as we move it in a parallel fashion toward the lower left is corner point ②. To find the coordinates of this point algebraically, we proceed as before by eliminating one of the variables from the two equations that intersect at this point (i.e., $5A + 10B = 45$ and $4A + 3B = 24$) so that we can solve for

FIGURE 2.9
Graphical Solution to the Holiday Meal Turkey Ranch Problem Using the Level Cost Line Method



the other. One way would be to multiply the first equation by 4, multiply the second equation by 5, and subtract the second equation from the first equation, as follows:

$$\begin{array}{lll} 4(5A + 10B = 45) & \text{implies} & 20A + 40B = 180 \\ -5(4A + 3B = 24) & \text{implies} & \underline{-(20A + 15B = 120)} \\ & & 25B = 60 \quad \text{implies } B = 2.40 \end{array}$$

Substituting $B = 2.40$ into the first equation yields $4A + (3)(2.40) = 24$, or $A = 4.20$. The cost at corner point ② is $\$0.10 \times 4.20 + \$0.15 \times 2.40 = \$0.78$. That is, Holiday Meal should use 4.20 pounds of brand A feed and 2.40 pounds of brand B feed, at a cost of \$0.78 per turkey per month. Observe that this solution has fractional values. In this case, however, this is perfectly logical because turkey feeds can be measured in fractional quantities.

As with the Flair Furniture example, we could also identify an optimal corner point in this problem by using the corner point method (i.e., evaluating the cost at all three identifiable corner points ①, ② , and ③).

2.6 Special Situations in Solving Linear Programming Problems

In each of the LP problems discussed so far, all the constraints in the model have affected the shape and size of the feasible region. Further, in each case, there has been a *single* corner point that we have been able to identify as the optimal corner point. There are, however, four special situations that may be encountered when solving LP problems: (1) redundant constraints, (2) infeasibility, (3) alternate optimal solutions, and (4) unbounded solutions. We illustrate the first three situations using the Flair Furniture example as the base model and the last one using the Holiday Meal Turkey Ranch example as the base model.

Redundant Constraints

A **redundant constraint** *is one that does not affect the feasible solution region.*

A **redundant constraint** is a constraint that does not affect the feasible region in any way. In other words, other constraints in the model are more restrictive and thereby negate the need to even consider the redundant constraint. The presence of redundant constraints is quite common in large LP models with many variables. However, it is typically impossible to determine whether a constraint is redundant just by looking at it.

Let's consider the LP model for the Flair Furniture problem again. Recall that the original model is

$$\text{Maximize profit} = \$7T + \$5C$$

subject to the constraints

$$\begin{aligned} 3T + 4C &\le 2{,}400 && \text{(carpentry time)} \\ 2T + 1C &\le 1{,}000 && \text{(painting time)} \\ C &\le 450 && \text{(maximum chairs allowed)} \\ T &\ge 100 && \text{(minimum tables required)} \\ T, C &\ge 0 && \text{(nonnegativity)} \end{aligned}$$

Now suppose that the demand for tables has become quite weak. Instead of specifying that at least 100 tables need to be made, the marketing department is now specifying that a *maximum* of 100 tables should be made. That is, the constraint should be $T \le 100$ instead of $T \ge 100$, as originally formulated. The revised feasible region for this problem due to this modified constraint is shown in Figure 2.10. From this figure, we see that the production limit constraints on chairs and tables are so restrictive that they make the carpentry and painting constraints redundant. That is, these two time constraints have no effect on the feasible region.

Infeasibility

Lack of a feasible solution region can occur if constraints conflict with one another.

Maximize $\$7T + \$5C$
subject to

$$\begin{aligned} 3T + 4C &\le 2{,}400 \\ 2T + 1C &\le 1{,}000 \\ C &\le 450 \\ T &\ge 100 \\ T, C &\ge 0 \end{aligned}$$

Infeasibility is a condition that arises when no single solution satisfies all of an LP problem's constraints. That is, no feasible solution region exists. Such a situation might occur, for example, if the problem has been formulated with conflicting constraints. As a graphical illustration of infeasibility, let us consider the Flair Furniture problem again (see the formulation in the margin note). Now suppose that Flair's marketing department has found that the demand for tables has become very strong. To meet this demand, it is now specifying that at least 600 tables should be made. That is, the constraint should now be $T \ge 600$ instead of $T \ge 100$. The revised graph for this problem due to this modified constraint is shown in Figure 2.11. From this figure, we see that there is no feasible solution region for this problem because of the presence of conflicting constraints.

Infeasibility is not uncommon in real-world, large-scale LP problems that involve hundreds of constraints. In such situations, the decision analyst coordinating the LP problem must resolve the conflict between the constraints causing the infeasibility and get them revised appropriately.

FIGURE 2.10
Problem with a Redundant Constraint

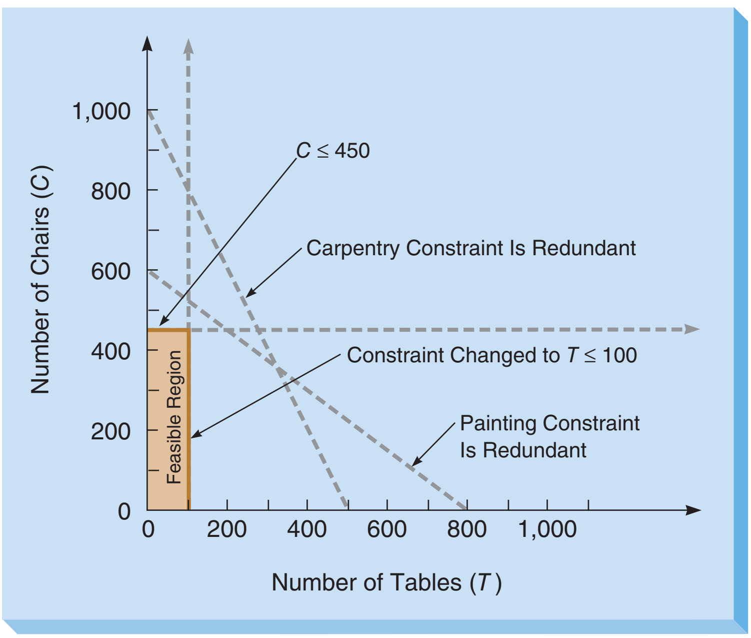
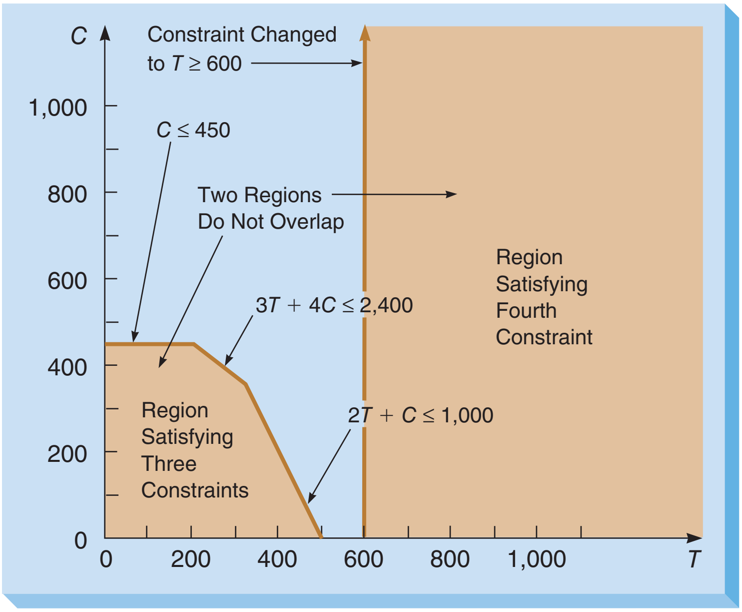


FIGURE 2.11
Example of an Infeasible Problem



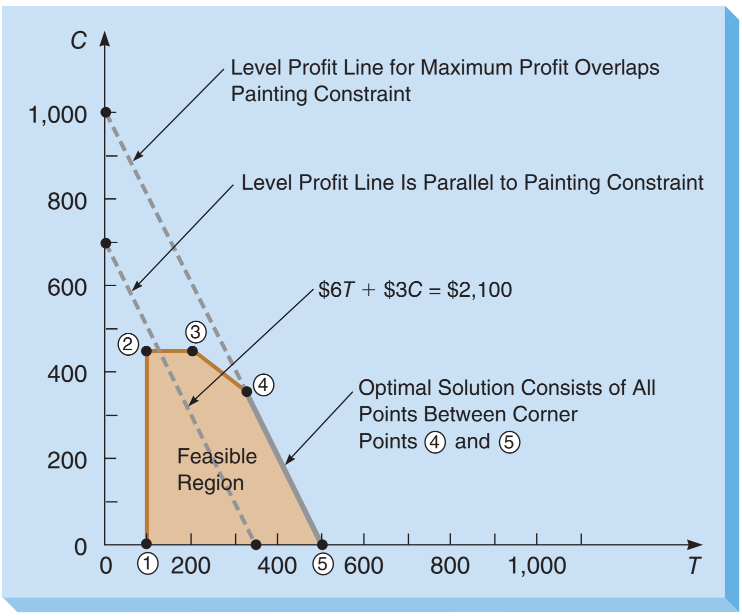
Alternate Optimal Solutions

Alternate optimal solutions are possible in LP problems.

An LP problem may, on occasion, have **alternate optimal solutions** (i.e., more than one optimal solution). Graphically, this is the case when the level profit (or cost) line runs parallel to a constraint in the problem that lies in the direction in which the profit (or cost) line is being moved—in other words, when the two lines have the same slope. To illustrate this situation, let us consider the Flair Furniture problem again.

Now suppose the marketing department has indicated that due to increased competition, profit contributions of both products have to be revised downward to \$6 per table and \$3 per chair. That is, the objective function is now $\$6T + \$3C$ instead of $\$7T + \$5C$. The revised graph for this problem is shown in Figure 2.12. From this figure, we note that the level profit line (shown here for a profit of \$2,100) runs parallel to the painting constraint equation. At a profit level of \$3,000, the level profit line will rest directly on top of this constraint line. This means that any point along the painting constraint equation between corner points ④ $(T = 320, C = 360)$ and ⑤ $(T = 500, C = 0)$ provides an optimal T and C combination.

FIGURE 2.12
Example of a Problem with Alternate Optimal Solutions



Far from causing problems, the presence of more than one optimal solution actually allows management greater flexibility in deciding which solution to select. The optimal objective function value remains the same at all alternate solutions.

Unbounded Solution

When a problem has an unbounded feasible region, it may not have a finite solution.

When an LP model has a bounded feasible region, as in the Flair Furniture example (i.e., it has an explicit boundary in every direction), it has an identifiable optimal corner point solution. However, if the feasible region is unbounded in one or more directions, as in the Holiday Meal Turkey Ranch example, depending on the objective function, the model may or may not have a finite solution. In the Holiday Meal problem, for example, we were able to identify a finite solution because the optimal corner point existed on the bounded side (refer to Figure 2.9 on page 36). However, what happens if the objective function is such that we have to move our level profit (or cost) lines away from the bounded side?

Minimize $\$0.10A + \$0.15B$
subject to

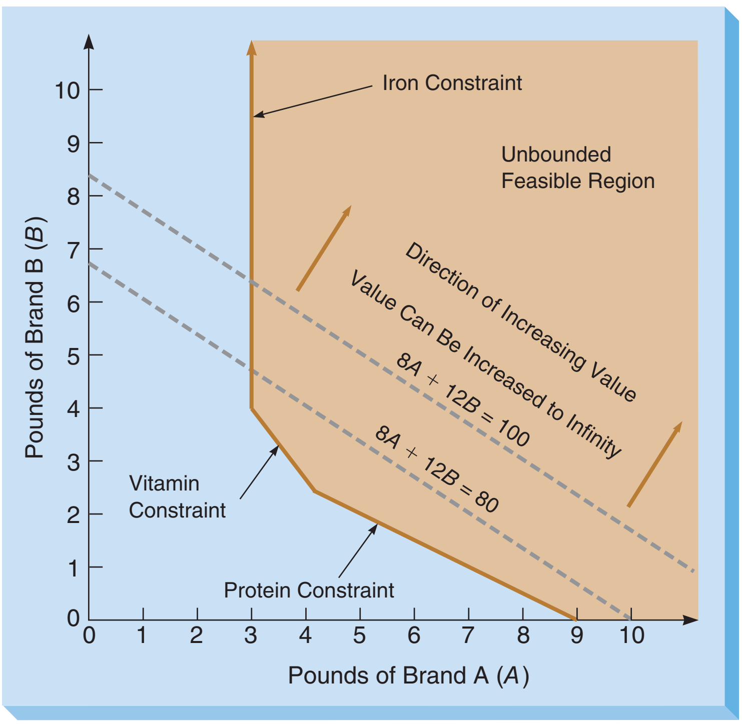
$$\begin{aligned} 5A + 10B &\geq 45 \\ 4A + 3B &\geq 24 \\ 0.5A &\geq 1.5 \\ A, B &\geq 0 \end{aligned}$$

To study this, let us again consider the Holiday Meal example (see the formulation in the margin note). Now suppose that instead of minimizing cost, the owner of the ranch wants to use a different objective function. Specifically, based on his experience with the feeds and their fattening impact on his turkeys, assume that the owner estimates that brand A feed yields a "fattening value" of 8 per pound, while brand B feed yields a fattening value of 12 per pound. The owner wants to find the diet that maximizes the total fattening value.

When the solution is unbounded in a maximization problem, the objective function value can be made infinitely large without violating any constraints.

The objective function now changes from "Minimize $\$0.10A + \$0.15B$" to "Maximize $8A + 12B$." Figure 2.13 shows the graph of this problem with the new objective function. As before, the feasible region (which has not changed) is unbounded, with three identifiable corner points on the bounded side. However, because this is now a maximization problem and the feasible region is unbounded in the direction in which profit increases, the solution itself is unbounded. That is, the profit can be made infinitely large without violating any constraints. In real-world situations, the occurrence of an **unbounded solution** usually means the problem has been formulated improperly. That is, either one or more constraints have the wrong sign or values, or some constraints have been overlooked. In Holiday Meal's case, it would indeed be wonderful to achieve an infinite fattening value, but that would have serious adverse implications for the amount of feed that the turkeys must eat each month!

FIGURE 2.13
Example of a Problem with an Unbounded Solution



IN ACTION Resource Allocation at Pantex

Companies often use optimization techniques such as Linear Programming to allocate limited resources to maximize profits or minimize costs. One of the most important resource allocation problems faced by the United States is dismantling old nuclear weapons and maintaining the safety, security, and reliability of the remaining systems. This problem is a primary concern of Pantex, which is responsible for disarming, evaluating, and maintaining the U.S. nuclear stockpile. The company is also responsible for storing critical weapons components that relate to U.S.–Russian nonproliferation agreements. Pantex constantly makes trade-offs in meeting the requirements of disarming some nuclear weapons versus maintaining existing nuclear weapons systems, while effectively allocating limited resources. Like many manufacturers, Pantex must allocate scarce resources among competing demands, all of which are important.

The team charged with solving the resource allocation problem at Pantex developed the Pantex Process Model (PPM). PPM is a sophisticated optimization system capable of analyzing nuclear needs over different time horizons. Since its development, PPM has become the primary tool for analyzing, planning, and scheduling issues at Pantex. PPM also helps to determine future resources. For example, it was used to gain government support for $17 million to modify an existing plant with new buildings and $70 million to construct a new plant.

Source: Based on E. Kjeldgaard et al. "Swords into Plowshares: Nuclear Weapon Dismantlement, Evaluation, and Maintenance at Pantex," *Interfaces* 30, (January–February, 2000): 57–82.

2.7 Setting Up and Solving Linear Programming Problems Using Excel's Solver

Although graphical solution approaches can handle LP models with only two decision variables, more complex solution procedures are necessary to solve larger LP models. Fortunately, such solution procedures exist. (We briefly discuss them in section 2.8.) However, rather than use these procedures to solve large LP models by hand, the focus in this textbook is on using Excel to set up and solve LP problems. Excel and other spreadsheet programs offer users the ability to analyze large LP problems by using built-in problem-solving tools.

Excel has a built-in solution tool for solving LP problems.

There are two main reasons why this textbook's focus on Excel for setting up and solving LP problems is logical and useful in practice:

- The use of spreadsheet programs is now very common, and virtually every organization has access to such programs.
- Because you are likely to be using Excel in many of your other courses, you are probably already familiar with many of its commands. Therefore, there is no need to learn any specialized software to set up and solve LP problems.

Excel uses an add-in named Solver to find the solution to LP-related problems. Solver is a Microsoft Excel add-in program that is available when you install Microsoft Office or Excel. The standard version of Solver that is included with Excel can handle LP problems with up to 200 decision variables and 100 constraints, not including simple lower and upper bounds on the decision variables (e.g., nonnegativity constraints). Larger versions of Solver are available for commercial use from Frontline Systems, Inc. (www.solver.com) which has developed and marketed this add-in for Excel (and other spreadsheet packages). We use Solver to solve LP problems in Chapters 2–5 and integer and nonlinear programming problems in Chapter 6.

The standard version of Solver is included with all versions of Excel.

Several other software packages (e.g., LINDO, GAMS) are capable of handling very large LP models. Although each program is slightly different in terms of its input and output formats, the approach each takes toward handling LP problems is basically the same. Hence, once you are experienced in dealing with computerized LP procedures, you can easily adjust to minor differences among programs.

Using Solver to Solve the Flair Furniture Problem

Recall that the decision variables T and C in the Flair Furniture problem denote the number of tables and chairs to make, respectively. The LP formulation for this problem is as follows:

$$\text{Maximize profit} = \$7T + \$5C$$

subject to the constraints

$$\begin{array}{llll} 3T + 4C & \leq & 2{,}400 & \text{(carpentry time)} \\ 2T + 1C & \leq & 1{,}000 & \text{(painting time)} \\ C & \leq & 450 & \text{(maximum chairs allowed)} \\ T & \geq & 100 & \text{(minimum tables required)} \\ T, C & \geq & 0 & \text{(nonnegativity)} \end{array}$$

Just as we discussed a three-step process to formulate an LP problem (i.e., decision variables, objective function, and constraints), setting up and solving a problem using Excel's Solver also involves three parts: changing variable cells, objective cell, and constraints. We discuss each of these parts in the following sections.

There is no prescribed layout for setting up LP problems in Excel.

In practice, there are no specific guidelines regarding the layout of an LP model in Excel. Depending on your personal preference and expertise, any model that satisfies the basic requirements discussed subsequently will work. However, for purposes of convenience and ease of explanation, we use (to the extent possible) the same layout for all problems in this textbook. Such a consistent approach is more suited to the beginning student of LP. As you gain experience with spreadsheet modeling of LP problems, we encourage you to try alternate layouts.

We represent all parameters associated with a decision variable in the same column.

In our suggested layout, we use a separate column to represent all the parameters (e.g., solution value, profit contribution, constraint coefficients) associated with each decision variable in the problem. The objective function and each constraint in the problem is then modeled on separate rows of the Excel worksheet. Although not required to solve the model, we also add several labels in our spreadsheet to make the entries as self-explanatory as possible.

Excel Note

The Companion Website for this textbook, at www.pearsonhighered.com/balakrishnan, contains the Excel file for each sample problem discussed here. The relevant file name appears in the margin next to each example.

Changing Variable Cells

Changing variable cells *are the decision variables in the problem.*

Solver refers to decision variables as **changing variable cells**. Each decision variable in a formulation is assigned to a unique cell in the spreadsheet. Although there are no rules regarding the relative positions of these cells, it is typically convenient to use cells that are next to each other.

WWW

File: 2-1.xls, sheet: 2-1A

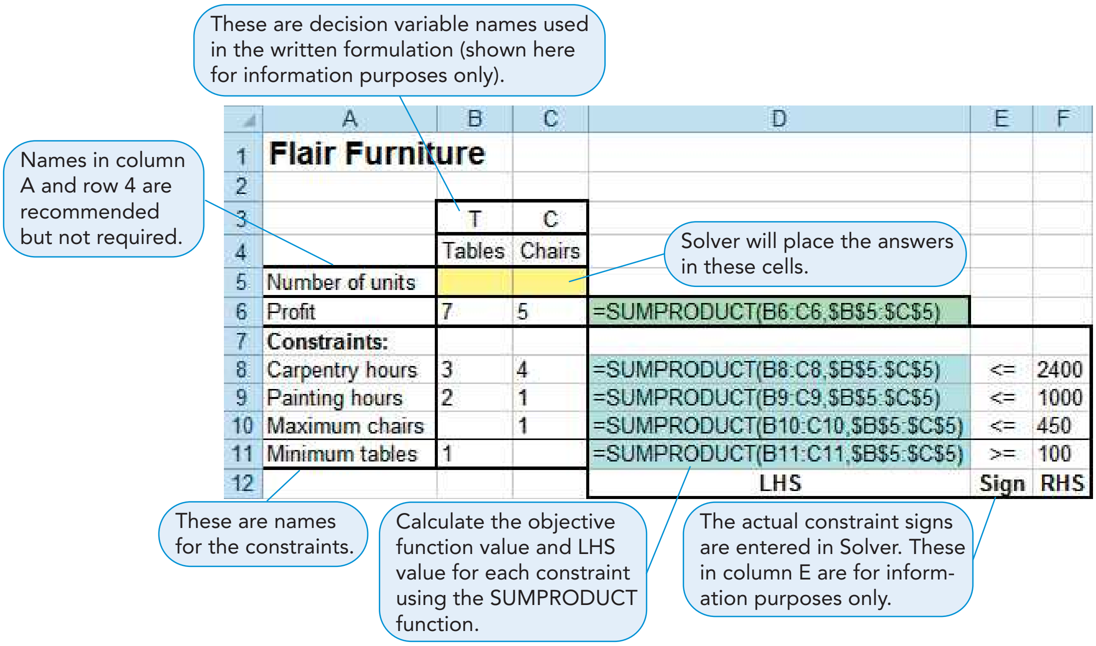
In the Flair Furniture example, two decision variables need to be assigned to any two cells in the spreadsheet. In Screenshot 2-1A, we use cells B5 and C5 to represent the number of tables to make (T) and the number of chairs to make (C), respectively.

The initial entries in these two cells can be blank or any value of our choice. At the conclusion of the Solver run, the optimal values of the decision variables will automatically be shown here (if an optimal solution is found).

It is possible, and often desirable, to format these cells using any of Excel's formatting features. For example, we can choose to specify how many decimal points to show for these values. Likewise, the cells can be assigned any name (instead of B5 and C5), using the naming option in Excel. Descriptive titles for these cells (such as those shown in cells A5, B4, and C4 of Screenshot 2-1A) are recommended to make the model as self-explanatory as possible, but they are not required to solve the problem.

Excel Notes

- In all our Excel layouts, for clarity, the changing variable cells (decision variables) are shaded yellow.
- In all our Excel layouts, we show the decision variable names (such as T and C) used in the written formulation of the model (see cells B3 and C3). These names have no role or relevance in using Solver to solve the model and can therefore be ignored. We show these decision variable names in our models in this textbook so that the equivalence of the written formulation and the Excel layout is clear.



	A	B	C	D	E	F
1	**Flair Furniture**					
2						
3		T	C			
4		Tables	Chairs			
5	Number of units					
6	Profit	7	5	=SUMPRODUCT(B6:C6,$B$5:$C$5)		
7	**Constraints:**					
8	Carpentry hours	3	4	=SUMPRODUCT(B8:C8,$B$5:$C$5)	<=	2400
9	Painting hours	2	1	=SUMPRODUCT(B9:C9,$B$5:$C$5)	<=	1000
10	Maximum chairs		1	=SUMPRODUCT(B10:C10,$B$5:$C$5)	<=	450
11	Minimum tables	1		=SUMPRODUCT(B11:C11,$B$5:$C$5)	>=	100
12				**LHS**	**Sign**	**RHS**

SCREENSHOT 2-1A
Formula View of the Excel Layout for Flair Furniture

The Objective Cell

The **objective cell** *contains the formula for the objective function.*

We can now set up the objective function, which Solver refers to as the **objective cell**. We select any cell in the spreadsheet (other than the cells allocated to the decision variables). In that cell, we enter the formula for the objective function, referring to the two decision variables by their cell references (B5 and C5 in this case). In Screenshot 2-1A, we use cell D6 to represent the objective function. Although we could use the unit profit contribution values ($7 per table and $5 per chair) directly in the formula, it is preferable to make the $7 and $5 entries in some cells in the spreadsheet and refer to them by their cell references in the formula in cell D6. This is a more elegant way of setting up the problem and is especially useful if subsequent changes in parameter values are necessary.

In Screenshot 2-1A, we have entered the 7 and 5 in cells B6 and C6, respectively. The formula in cell D6 can therefore be written as

=B6*B5+C6*C5

The = at the start of the equation lets Excel know that the entry is a formula. This equation corresponds exactly to the objective function of the Flair Furniture problem. If we had left cells B5 and C5 blank, the result of this formula would initially be shown as 0. As with cells B5 and C5, we can format the objective cell (D6) in any manner. For example, because D6 denotes the profit, in dollars, earned by Flair Furniture, we can format it to show the result as a dollar value.

Excel's SUMPRODUCT function makes it easy to enter even long expressions.

If there are several decision variables in a problem, however, formulas can become somewhat long, and typing them can become quite cumbersome. In such cases, you can use Excel's SUMPRODUCT function to express the equation efficiently. The syntax for the SUMPRODUCT function requires specifying two cell ranges of equal size, separated by a comma.[5] One of the ranges defines the cells containing the profit contributions (cells B6:C6), and the other defines the cells containing the decision variables (cells B5:C5). The SUMPRODUCT function computes the products of the first entries in each range, second entries in each range, and so on. It then sums these products.

Based on the preceding discussion, as shown in Screenshot 2-1A, the objective function for Flair Furniture can be expressed as

=SUMPRODUCT(B6:C6,$B$5:$C$5)

[5] The SUMPRODUCT function can also be used with more than two cell ranges. See Excel's help feature for more details on this function.

Note that this is equivalent to =B6*B5+C6*C5. Also, the use of the $ symbol while specifying the cell references (in the second cell range) keeps those cell references fixed in the formula when we copy this cell to other cells. This is especially convenient because, as we show next, the formula for each constraint in the model also follows the same structure as the objective function.

Excel Note

In each of our Excel layouts, for clarity, the objective cell (objective function) has been shaded green.

Constraints

Constraints in Solver include three entries: LHS, RHS, and sign.

We must now set up each constraint in the problem. To achieve this, let us first separate each constraint into three parts: (1) a *left-hand-side (LHS)* part consisting of every term to the left of the equality or inequality sign, (2) a *right-hand-side (RHS)* part consisting of all terms to the right of the equality or inequality sign, and the (3) equality or inequality sign itself. The RHS in most cases may just be a fixed number—that is, a constant.

CREATING CELLS FOR CONSTRAINT LHS VALUES We now select a unique cell for each **constraint LHS** in the formulation (one for each constraint) and type in the relevant formula for that constraint. As with the objective function, we refer to the decision variables by their cell references. In Screenshot 2-1A, we use cell D8 to represent the LHS of the carpentry time constraint. We have entered the coefficients (i.e., 3 and 4) on the LHS of this constraint in cells B8 and C8, respectively. Then, either of the following formulas would be appropriate in cell D8:

=B8*B5+C8*C5

or

=SUMPRODUCT(B8:C8,$B$5:$C$5)

Here again, the SUMPRODUCT function makes the formula compact in situations in which the LHS has many terms. Note the similarity between the objective function formula in cell D6 [=SUMPRODUCT(B6:C6,$B$5:$C$5)] and the LHS formula for the carpentry constraint in cell D8 [=SUMPRODUCT(B8:C8,$B$5:$C$5)]. In fact, because we have anchored the cell references for the decision variables (B5 and C5) using the $ symbol in cell D6, we can simply copy the formula in cell D6 to cell D8.

Formula in cell D9: =SUMPRODUCT(B9:C9, $B$5:$C$5

The LHS formula for the painting hours constraint (cell D9), chairs production limit constraint (cell D10), and tables minimum production constraint (cell D11) can similarly be copied from cell D6. As you have probably recognized by now, the LHS cell for virtually every constraint in an LP formulation can be created in this fashion.

Excel Note

In each of our Excel layouts, for clarity, cells denoting LHS formulas of constraints have been shaded blue.

In Solver, the RHS of a constraint can also include a formula.

CREATING CELLS FOR CONSTRAINT RHS VALUES When all the LHS formulas have been set up, we can pick unique cells for each **constraint RHS** in the formulation. Although the Flair Furniture problem has only constants (2,400, 1,000, 450, and 100, respectively) for the four constraints, it is perfectly valid in Solver for the RHS to also have a formula like the LHS. In Screenshot 2-1A, we show the four RHS values in cells F8:F11.

The actual sign for each constraint is entered directly in Solver.

CONSTRAINT TYPE In Screenshot 2-1A, we also show the sign ($\leq$, $\geq$, or $=$) of each constraint between the LHS and RHS cells for that constraint (see cells E8:E11). Although this makes each constraint easier to understand, note that the inclusion of these signs here is for information purposes only. As we show next, the actual sign for each constraint is entered directly in Solver.

NONNEGATIVITY CONSTRAINTS It is not necessary to specify the nonnegativity constraints (i.e., $T \geq 0$ and $C \geq 0$) in the model using the previous procedure. As we will see shortly, there is a simple option available in Solver to automatically enforce these constraints.

Entering Information in Solver

After all the constraints have been set up, we invoke the Solver Parameters window in Excel by clicking the Data tab and then selecting Solver in the Analysis group, as shown in Screenshot 2-1B(a).[6] The Solver Parameters window is shown in Screenshot 2-1B(b).

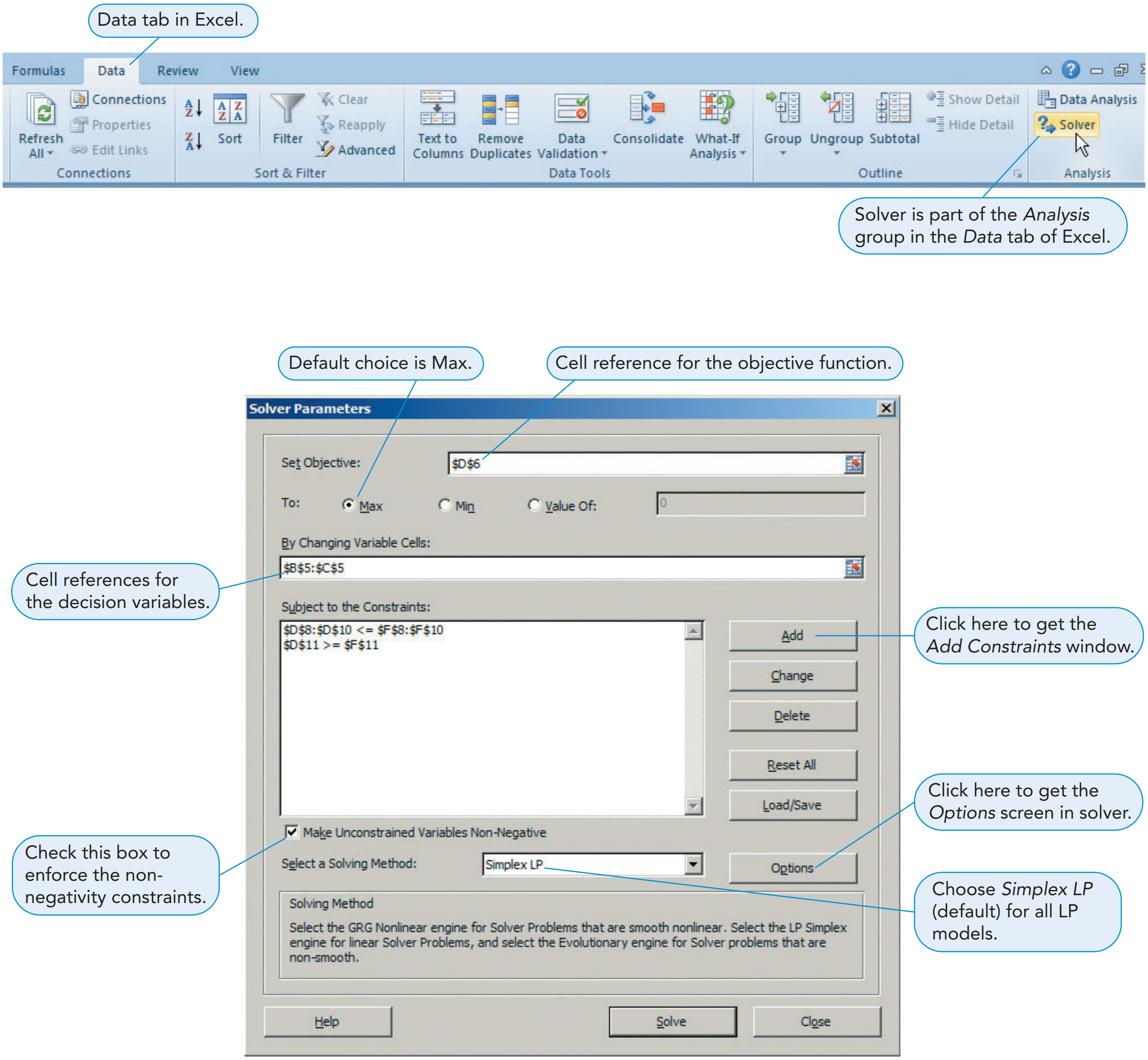
The default in Solver is to maximize the objective cell.

SPECIFYING THE OBJECTIVE CELL We first enter the relevant cell reference (i.e., cell D6) in the Set Objective box. The default in Solver is to maximize the objective value. (Note that the Max option is already selected.) For a minimization problem, we must click the Min option to specify that the objective function should be minimized. The third option (Value Of) allows

SCREENSHOT 2-1B
Solver Parameters Window for Flair Furniture

(a)

(b)



[6] If you do not see Solver in the Analysis group within the Data tab in Excel, refer to Appendix B, section B.6, *Installing and Enabling Excel Add-Ins*, for instructions on how to fix this problem. Alternatively, type *Solver* in Excel's help feature and select Load the Solver Add-in for detailed instructions.

us to specify a value that we want the objective cell to achieve, rather than obtain the optimal solution. (We do not use this option in our study of LP and other mathematical programming models.)

Changing variable cells can be entered as a block or as individual cell references separated by commas.

SPECIFYING THE CHANGING VARIABLE CELLS We now move the cursor to the box labeled By Changing Variable Cells. We enter the cell references for the decision variables in this box. If the cell references are next to each other, we can simply enter them as one block. For example, we could enter B5:C5 for Flair Furniture's problem. (If we use the mouse or keyboard to highlight and select cells B5 and C5, Excel automatically puts in the $ anchors, as shown in Screenshot 2-1B.) If the cells are not contiguous (i.e., not next to each other), we can enter the changing variable cells by placing a comma between noncontiguous cells (or blocks of cells). So, for example, we could enter B5,C5 in the By Changing Variable Cells window for this specific problem.

The Add Constraint window is used to enter constraints.

SPECIFYING THE CONSTRAINTS Next, we move to the box labeled Subject to the Constraints and click the Add button to enter the relevant cell references for the LHS and RHS of each constraint. The Add Constraint window (shown in Screenshot 2-1C) has a box titled Cell Reference in which we enter the cell reference of the constraint's LHS, a drop-down menu in which we specify the constraint's sign, and a second box titled Constraint in which we enter the cell reference of the constraint's RHS. The drop-down menu has six choices: $\leq$, $\geq$, $=$, Int (for integer), Bin (for binary), and dif (for all different). (We discuss the last three choices in Chapter 6.)

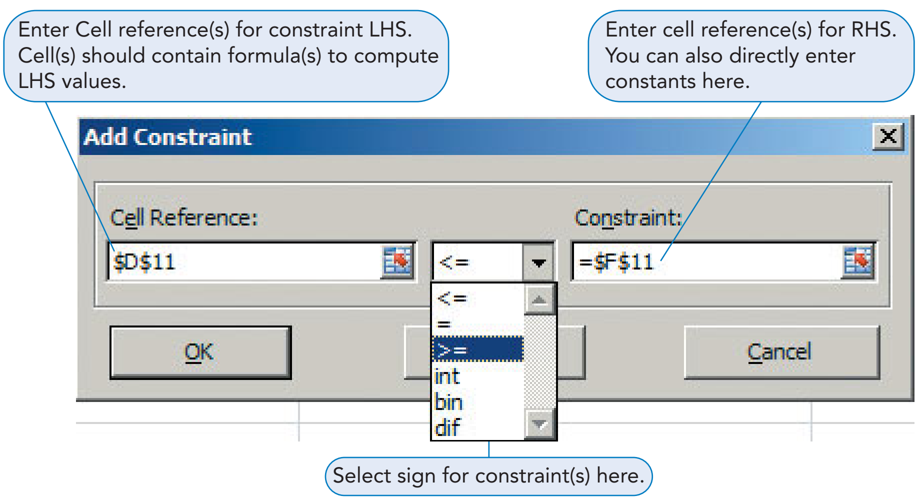
We can either add constraints one at a time or add blocks of constraints that have the same sign ($\leq$, $\geq$, or $=$) at the same time. For instance, we could first add the carpentry constraint by entering D8 in the LHS input box, entering F8 in the RHS input box, and selecting the $\leq$ sign from the drop-down menu. As noted earlier, the $\leq$ sign shown in cell E8 is not relevant in Solver, and we must enter the sign of each constraint by using the Add Constraint window. We can now add the painting constraint by entering D9 and F9 in the LHS and RHS input boxes, respectively. Next, we can add the chairs limit constraint by entering D10 and F10 in the LHS and RHS input boxes, respectively. Finally, we can add the minimum table production constraint by entering D11 and F11 in the LHS and RHS input boxes, respectively. Note that in this constraint's case, we should select the $\geq$ sign from the drop-down menu.

Constraints with the same sign can be entered as a block.

Alternatively, because the first three constraints have the same sign ($\leq$), we can input cells D8 to D10 in the LHS input box (i.e., enter D8:D10) and correspondingly enter F8:F10 in the RHS input box. We select $\leq$ as the sign between these LHS and RHS entries. Solver interprets this as taking each entry in the LHS input box and setting it $\leq$ to the corresponding entry in the RHS input box (i.e., D8 $\leq$ F8, D9 $\leq$ F9, and D10 $\leq$ F10).

Using the latter procedure, note that it is possible to have just three entries in the constraints window: one for all the $\leq$ constraints in the model, one for all the $\geq$ constraints in the model, and one for all the $=$ constraints in the model. This, of course, requires that the spreadsheet layout be such that the LHS and RHS cells for all constraints that have the same sign are in

SCREENSHOT 2-1C
Solver Add Constraint Window



contiguous blocks, as in Screenshot 2-1A. However, as we demonstrate in several examples in Chapter 3, this is quite easy to do.

At any point during or after the constraint input process, we can use the Change or Delete buttons in the Subject to the Constraints box to modify one or more constraints, as necessary. It is important to note that we *cannot* enter the formula for the objective function and the LHS and/or RHS of constraints from within the Solver Parameters window. The formulas must be created in appropriate cells in the spreadsheet before using the Solver Parameters window. Although it is possible to directly enter constants (2,400, 1,000, 450, and 100 in our model) in the RHS input box while adding constraints, it is preferable to make the RHS also a cell reference (F8, F9, F10, and F11 in our model).

Check the **Make Unconstrained Variables Non-Negative** *box in Solver to enforce the nonnegativity constraints.*

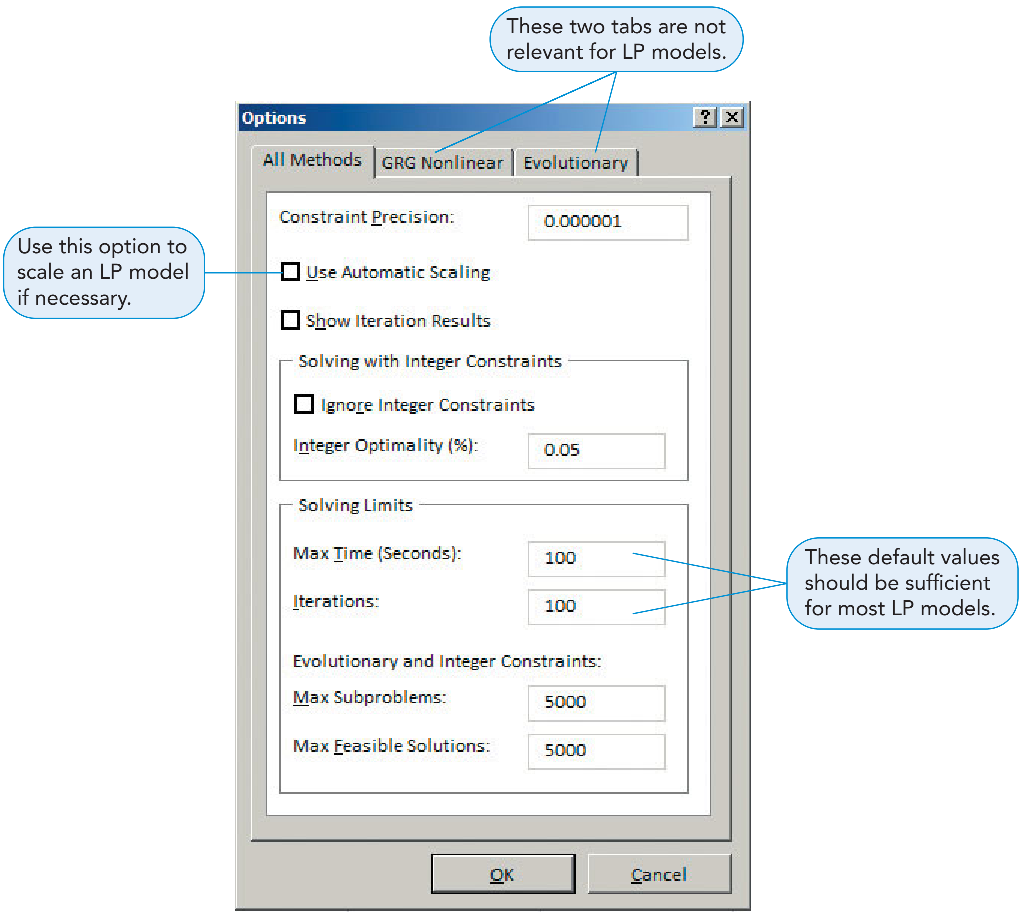
SPECIFYING THE NONNEGATIVITY CONSTRAINTS Directly below the box labeled Subject to the Constraints (see Screenshot 2-1B), there is a box labeled Make Unconstrained Variables Non-Negative. This box is checked by default in Excel; for most LP models, it should remain this way. The checked box automatically enforces the nonnegativity constraint for all the decision variables in the model.

Select **Simplex LP** *as the solving method in Solver.*

SOLVING METHOD Next, we move to the box labeled Select a Solving Method. To solve LP problems, we should leave this option at its default setting of Simplex LP. Selecting this setting directs Solver to solve LP models efficiently and provide a detailed Sensitivity Report, which we cover in Chapter 4. Clicking the down arrow in this box reveals two other method choices: GRG Nonlinear and Evolutionary. We will discuss the GRG Nonlinear procedure in Chapter 6.

SOLVER OPTIONS After all constraints have been entered, we are ready to solve the model. However, before clicking the Solve button on the Solver Parameters window, we click the Options button to open the Solver Options window (shown in Screenshot 2-1D) and focus on the choices available in the All Methods tab. (The options in the GRG Nonlinear and Evolutionary tabs are not relevant for LP models.) For solving most LP problems, we do not have to change

SCREENSHOT 2-1D
Solver Options Window



any of the default parameters for these options. The defaults of 100 seconds and 100 iterations should be adequate. The options related to Evolutionary and Integer Constraints are not relevant for LP models. To see details of each iteration taken by Solver to go from the initial solution to the optimal solution (if one exists), we can check the Show Iterations Results box.

It is a good idea to scale coefficient values in LP models.

With regard to the option called Use Automatic Scaling (see Screenshot 2-1D), it is a good idea in practice to scale problems in which values of the objective function coefficients and constraint coefficients of different constraints differ by several orders of magnitude. For instance, a problem in which some coefficients are in millions while others have fractional values would be considered a poorly scaled model. Due to the effects of a computer's finite precision arithmetic, such poorly scaled models could cause difficulty for Solver, leading to fairly large rounding errors. Checking the automatic scaling box directs Solver to scale models that it detects as poorly scaled and possibly avoid such rounding problems.

WWW

File: 2-1.xls, sheet: 2-1E

SOLVING THE MODEL When the Solve button is clicked, Solver executes the model and displays the results, as shown in Screenshot 2-1E. Before looking at the results, it is important to read the message in the Solver Results window to verify that Solver found an optimal solution. In some cases, the window indicates that Solver is unable to find an optimal solution (e.g., when the formulation is infeasible or the solution space is unbounded). Table 2.2 shows several different Solver messages that could result when an LP model is solved, the meaning of each message, and a possible cause for each message.

Solver provides options to obtain different reports.

The Solver Results window also indicates that there are three reports available: Answer, Sensitivity, and Limits. We discuss the Answer Report in the next section and the Sensitivity Report in Chapter 4. The Limits Report is not useful for our discussion here, and we therefore ignore it. Note that in order to get these reports, we must select them by clicking the relevant report names to highlight them before clicking OK on the Solver Results window.

SCREENSHOT 2-1E
Excel Layout and Solver Solution for Flair Furniture (Solver Results Window Also Shown)

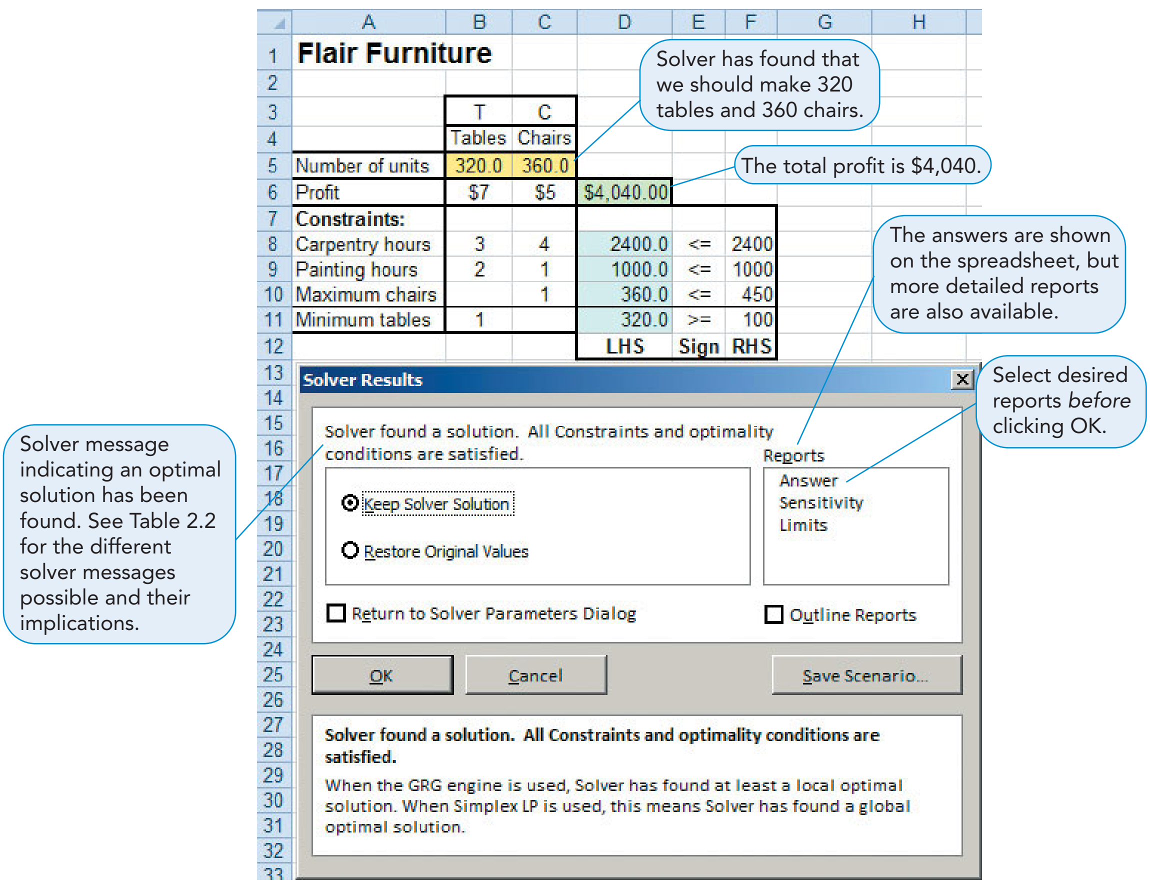


TABLE 2.2
Possible Messages in the Solver Results Window

MESSAGE	MEANING	POSSIBLE CAUSE
Solver found a solution. All Constraints and optimality conditions are satisfied.	Ideal message!	*Note:* This does *not* mean the formulation and/or solution is correct. It just means there are no syntax errors in the Excel formulas and Solver entries.
Solver could not find a feasible solution.	There is no feasible region.	Incorrect entries in LHS formulas, signs, and/or RHS values of constraints.
The Objective Cell values do not converge.	Unbounded solution.	Incorrect entries in LHS formulas, signs, and/or RHS values of constraints.
Solver encountered an error value in the Objective Cell or a Constraint cell.	Formula error in the objective cell or a constraint cell. At least one of the cells in the model becomes an error value when Solver tries different values for the changing variable cells.	Most common cause is division by zero in some cell.
The linearity conditions required by this LP Solver are not satisfied.	The Simplex LP method has been specified in Solver to solve this model, but one or more formulas in the model are not linear.	Multiplication or division involving two or more variables in some cell. *Note:* Solver sometimes gives this error message even when the formulas are linear. This occurs especially when both the LHS and RHS of a constraint have formulas. In such cases, we should manipulate the constraint algebraically to make the RHS a constant.

Cells B5 and C5 show the optimal quantities of tables and chairs to make, respectively, and cell D6 shows the optimal profit. Cells D8 to D11 show the LHS values of the four constraints. For example, cell D8 shows the number of carpentry hours used.

The Answer Report presents the results in a more detailed manner.

ANSWER REPORT If requested, Solver provides the **Answer Report** in a separate worksheet. The report for Flair's problem is shown in Screenshot 2-1F. (We have added grid lines to this report to make it clearer.) The report essentially provides the same information as that discussed previously but in a more detailed and organized manner. In addition to showing the initial and final (optimal) values for the objective function and each decision variable, it includes a column titled Integer, which indicates whether the decision variable was specified as continuous valued or integer valued in the model. (We will discuss integer valued variables in Chapter 6). The report also includes the following information for each constraint in the model:

File: 2-1.xls, sheet: 2-1F

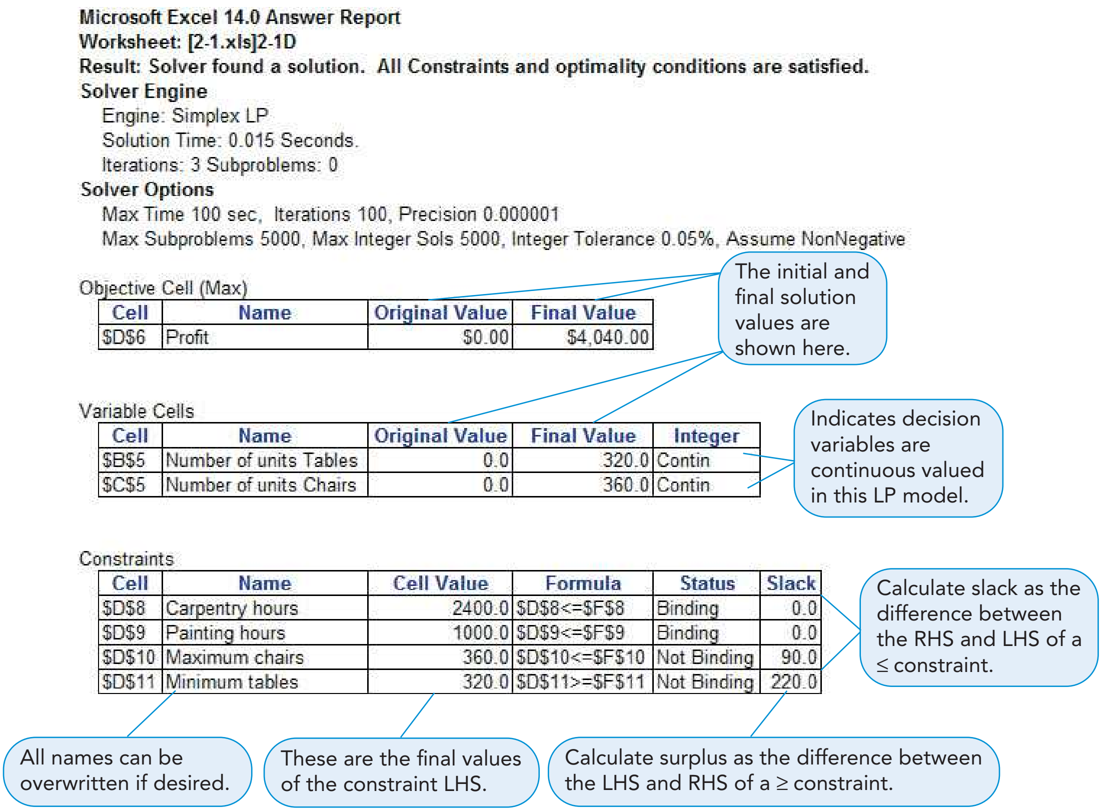
Names in Solver reports can be edited, if desired.

1. Cell. Cell reference corresponding to the LHS of the constraint. For example, cell D8 contains the formula for the LHS of the carpentry constraint.
2. Name. Descriptive name of the LHS cell. We can use Excel's naming feature to define a descriptive name for any cell (or cell range) simply by typing the desired name in the Name box (which is at the left end of the formula bar in any Excel worksheet and has the cell reference listed by default). If we do so, the cell name is reported in this column. If no name is defined for a cell, Solver extracts the name shown in this column from the information provided in the spreadsheet layout. Solver simply combines labels (if any) to the left of and above the LHS cell to create the name for that cell. Note that these labels can be overwritten manually, if necessary. For example, the name Profit for the objective cell (cell D6) can be overwritten to say Total Profit. Observe that the Excel layout we have used here ensures that all names automatically generated by Solver are logical.
3. Cell Value. The final value of the LHS of the constraint at the optimal solution. For example, the cell value for the carpentry time constraint indicates that we are using 2,400 hours at the optimal solution.
4. Formula. The formula specified in Solver for the constraint. For example, the formula entered in Solver for the carpentry time constraint is $D8 \leq F8$.

Binding *means the constraint is exactly satisfied and LHS = RHS.*

5. Status. Indicates whether the constraint is binding or nonbinding. *Binding* means that the constraint becomes an equality (i.e., LHS $=$ RHS) at the optimal solution. For a $\leq$

SCREENSHOT 2-1F
Solver's Answer Report for Flair Furniture



Microsoft Excel 14.0 Answer Report
Worksheet: [2-1.xls]2-1D
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.015 Seconds.
Iterations: 3 Subproblems: 0
Solver Options
Max Time 100 sec, Iterations 100, Precision 0.000001
Max Subproblems 5000, Max Integer Sols 5000, Integer Tolerance 0.05%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
\$D\$6	Profit	\$0.00	\$4,040.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
\$B\$5	Number of units Tables	0.0	320.0	Contin
\$C\$5	Number of units Chairs	0.0	360.0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$D\$8	Carpentry hours	2400.0	\$D\$8<=\$F\$8	Binding	0.0
\$D\$9	Painting hours	1000.0	\$D\$9<=\$F\$9	Binding	0.0
\$D\$10	Maximum chairs	360.0	\$D\$10<=\$F\$10	Not Binding	90.0
\$D\$11	Minimum tables	320.0	\$D\$11>=\$F\$11	Not Binding	220.0

constraint, this typically means that all the available amounts of that resource are fully used in the optimal solution. In Flair's case, the carpentry and painting constraints are both binding because we are using all the available hours in either case.

For a $\geq$ constraint, *binding* typically means we are exactly satisfying the minimum level required by that constraint. In Flair's case, the minimum tables required constraint is nonbinding because we plan to make 320 as against the required minimum of 100.

Slack typically refers to the amount of unused resource in a $\leq$ constraint.

6. Slack. Magnitude (absolute value) of the difference between the RHS and LHS values of the constraint. Obviously, if the constraint is binding, **slack** is zero (because LHS = RHS). For a nonbinding $\leq$ constraint, slack typically denotes the amount of resource that is left unused at the optimal solution. In Flair's case, we are allowed to make up to 450 chairs but are planning to make only 360. The absolute difference of 90 between the RHS and LHS $(= |450 - 360|)$ is the slack in this constraint.

Surplus typically refers to the amount of oversatisfaction of a $\geq$ constraint.

For a nonbinding $\geq$ constraint, we call this term **surplus** (even though Solver refers to this difference in all cases as slack). A surplus typically denotes the extent to which the $\geq$ constraint is oversatisfied at the optimal solution. In Flair's case, we are planning to make 320 tables even though we are required to make only 100. The absolute difference of 220 between the RHS and LHS $(= |100 - 320|)$ is the surplus in this constraint.

Using Solver to Solve Flair Furniture Company's Modified Problem

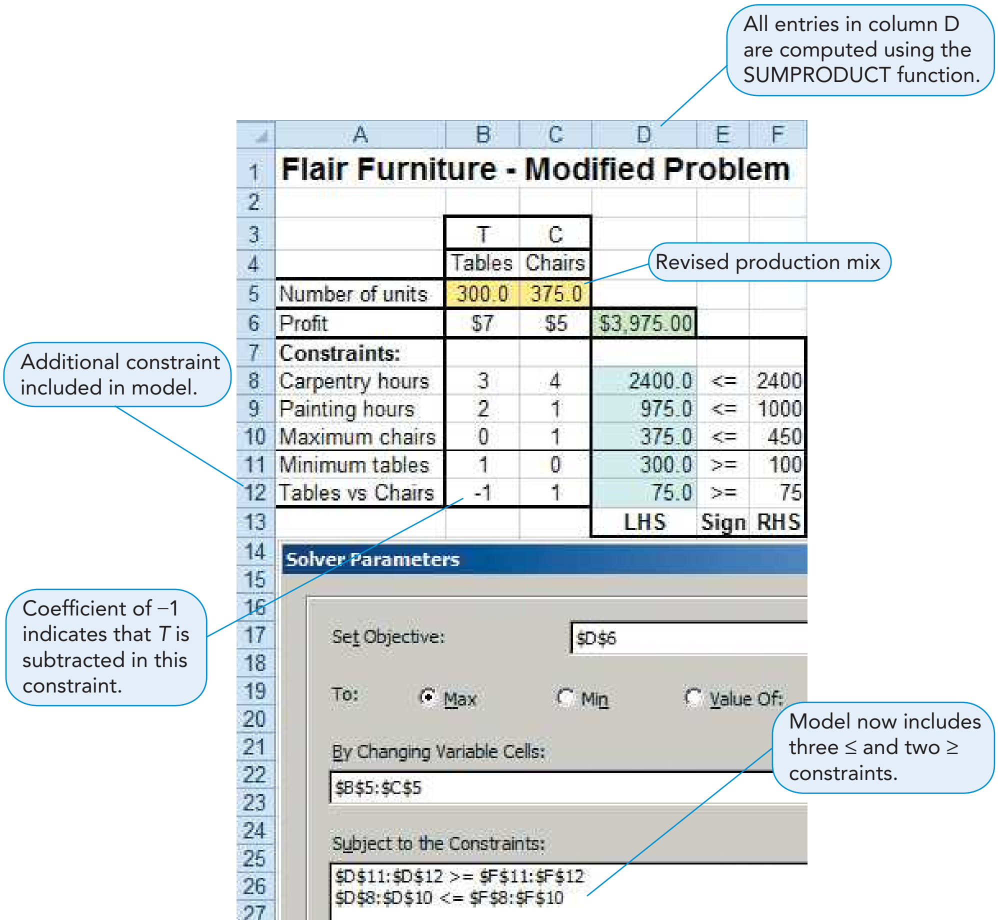
Recall that after solving Flair Furniture's problem using a graphical approach, we added a new constraint specified by the marketing department. Specifically, we needed to ensure that the number of chairs made this month is at least 75 more than the number of tables made. The constraint was expressed as

$$C - T \geq 75$$

File: 2-2.xls

The Excel layout and Solver entries for Flair's modified problem are shown in Screenshot 2-2. Note that the constraint coefficient for T is entered as –1 in cell B12 to reflect the fact that the

SCREENSHOT 2-2
Excel Layout and Solver Entries for Flair Furniture—Revised Problem



variable T is subtracted in the expression. The formula in cell D12 is the same SUMPRODUCT function used in cells D8:D11. The optimal solution now is to make 300 tables and 375 chairs, for a profit of \$3,975, the same solution we obtained graphically in Figure 2.7 on page 33.

Using Solver to Solve the Holiday Meal Turkey Ranch Problem

Now that we have studied how to set up and solve a maximization LP problem using Excel's Solver, let us consider a minimization problem—the Holiday Meal Turkey Ranch example. Recall that the decision variables A and B in this problem denote the number of pounds of brand A feed and brand B feed to use per month, respectively. The LP formulation for this problem is as follows:

$$\text{Minimize cost} = \$0.10A + \$0.15B$$

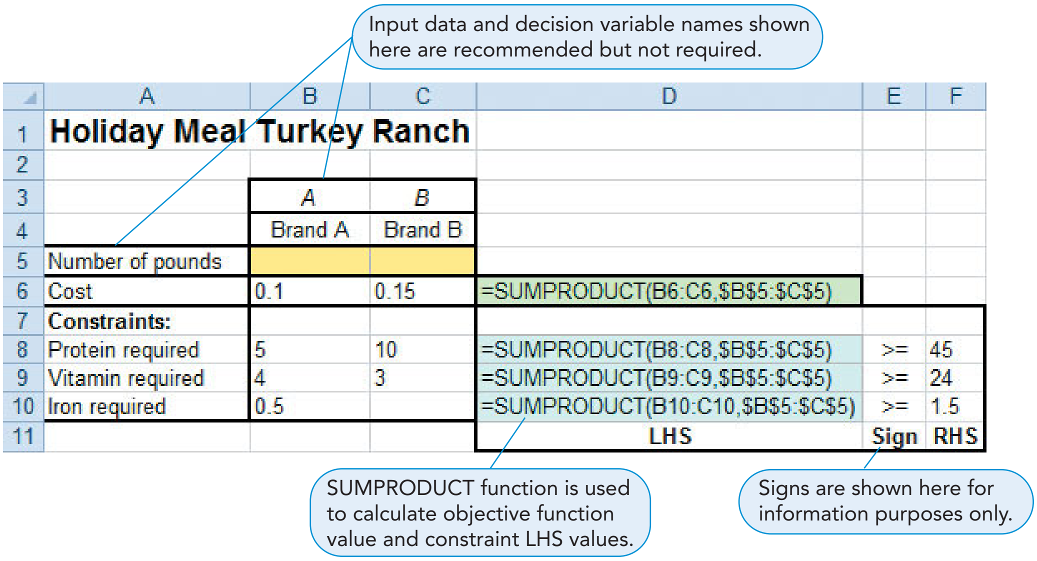
subject to the constraints

$$\begin{aligned} 5A + 10B &\geq 45 && \text{(protein required)} \\ 4A + 3B &\geq 24 && \text{(vitamin required)} \\ 0.5A &\geq 1.5 && \text{(iron required)} \\ A, B &\geq 0 && \text{(nonnegativity)} \end{aligned}$$

WWW
File: 2-3.xls

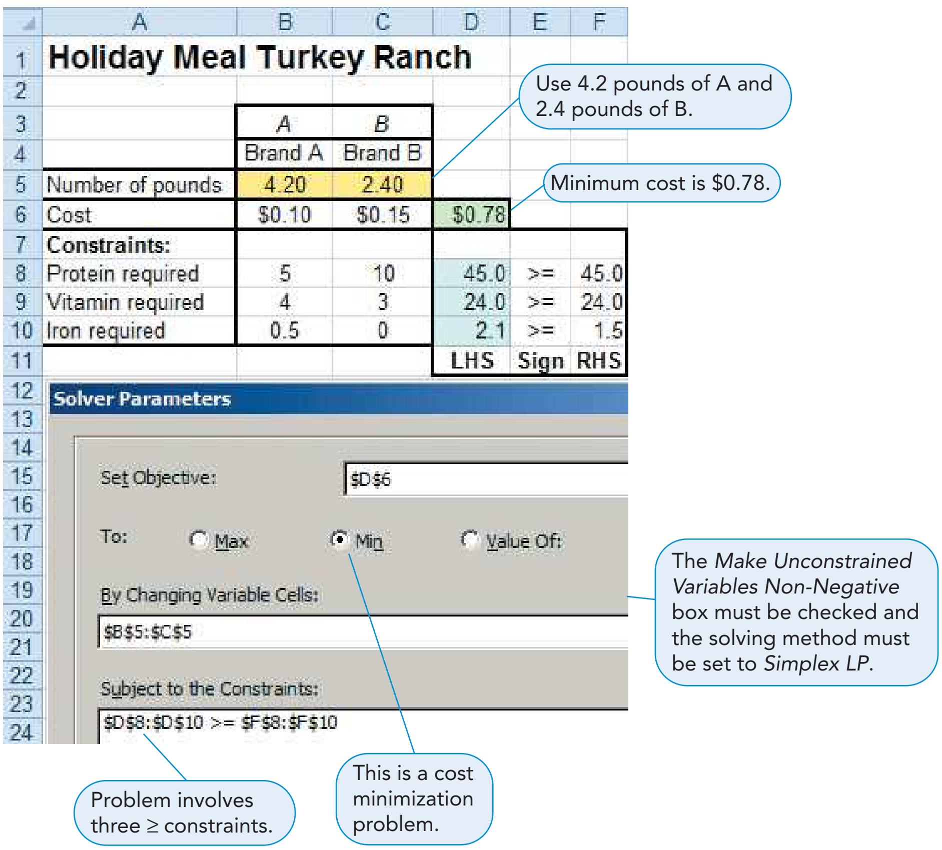
The formula view of the Excel layout for the Holiday Meal Turkey Ranch LP problem is shown in Screenshot 2-3A. The solution values and the Solver Parameters window are shown in Screenshot 2-3B. Note that Solver shows the problem as being solved as a Min problem. As with the Flair Furniture example, all problem parameters are entered as entries in different cells of the spreadsheet, and Excel's SUMPRODUCT function is used to compute the objective function as well as the LHS values for all three constraints (corresponding to protein, vitamin, and iron).

SCREENSHOT 2-3A
Formula View of the Excel Layout for Holiday Meal



	A	B	C	D	E	F
1	Holiday Meal Turkey Ranch					
2						
3		A	B			
4		Brand A	Brand B			
5	Number of pounds					
6	Cost	0.1	0.15	=SUMPRODUCT(B6:C6,$B$5:$C$5)		
7	Constraints:					
8	Protein required	5	10	=SUMPRODUCT(B8:C8,$B$5:$C$5)	>=	45
9	Vitamin required	4	3	=SUMPRODUCT(B9:C9,$B$5:$C$5)	>=	24
10	Iron required	0.5		=SUMPRODUCT(B10:C10,$B$5:$C$5)	>=	1.5
11				LHS	Sign	RHS

SCREENSHOT 2-3B
Excel Layout and Solver Entries for Holiday Meal



	A	B	C	D	E	F
1	Holiday Meal Turkey Ranch					
2						
3		A	B			
4		Brand A	Brand B			
5	Number of pounds	4.20	2.40			
6	Cost	$0.10	$0.15	$0.78		
7	Constraints:					
8	Protein required	5	10	45.0	>=	45.0
9	Vitamin required	4	3	24.0	>=	24.0
10	Iron required	0.5	0	2.1	>=	1.5
11				LHS	Sign	RHS

As expected, the optimal solution is the same as the one we obtained using the graphical approach. Holiday Meal should use 4.20 pounds of brand A feed and 2.40 pounds of brand B feed, at a cost of $0.78 per turkey per month. The protein and vitamin constraints are binding at the optimal solution. However, we are providing 2.1 units of iron per turkey per month even though we are required to provide only 1.5 units (i.e., an oversatisfaction, or *surplus*, of 0.6 units).

2.8 Algorithmic Solution Procedures for Linear Programming Problems

Simplex Method

So far, we have looked at examples of LP problems that contain only two decision variables. With only two variables, it is possible to use a graphical approach. We plotted the feasible region and then searched for an optimal corner point and corresponding profit or cost. This approach provides a good way to understand the basic concepts of LP. Most real-life LP problems, however, have more than two variables and are thus too large for the simple graphical solution procedure. Problems faced in business and government can have dozens, hundreds, or even thousands of variables. We need a more powerful method than graphing; for this we turn to a procedure called the **simplex method**.

Recall that the theory of LP states that the optimal solution will lie at a corner point of the feasible region. In large LP problems, the feasible region cannot be graphed because it has many dimensions, but the concept is the same.

How does the simplex method work? The concept is simple and similar to graphical LP in one important respect: In graphical LP, we examine each of the corner points; LP theory tells us that an optimal solution lies at one of them. In LP problems containing several variables, we may not be able to graph the feasible region, but an optimal solution still lies at a corner point of the many-sided, many-dimensional figure (called an n-dimensional polyhedron) that represents the area of feasible solutions. The simplex method examines the corner points in a systematic fashion, using basic algebraic concepts. It does so as an **iterative process**—that is, repeating the same set of steps time after time until an optimal solution is reached. Each iteration of the simplex method brings a value for the objective function that is no worse (and usually better) than the current value. Hence, we progressively move closer to an optimal solution.

The* simplex method *systematically examines corner points, using algebraic steps, until an optimal solution is found.

In most software packages, including Excel's Solver, the simplex method has been coded in a very efficient manner to exploit the computational capabilities of modern computers. As a result, for most LP problems, the simplex method identifies an optimal corner point after examining just a tiny fraction of the total number of corner points in the feasible region.

Karmarkar's Algorithm

Karmarkar's method follows a path of points inside the feasible region.

In 1984, Narendra Karmarkar developed an alternative to the simplex algorithm. The new method, called Karmarkar's algorithm, often takes significantly less computer time to solve very large LP problems.[7]

Whereas the simplex algorithm finds a solution by moving from one adjacent corner point to the next, following the outside edges of the feasible region, Karmarkar's method follows a path of points on the inside of the feasible region. Karmarkar's method is also unique in its ability to handle an extremely large number of constraints and variables, thereby giving LP users the capacity to solve previously unsolvable problems.

Although it is likely that the simplex method will continue to be used for many LP problems, a newer generation of LP software has been built around Karmarkar's algorithm.

Summary

In this chapter we introduce a mathematical modeling technique called linear programming (LP). Analysts use LP models to find an optimal solution to problems that have a series of constraints binding the objective value. We discuss how to formulate LP models and then show how models with only two decision variables can be solved graphically. The graphical solution approach of this chapter provides a conceptual basis for tackling larger, more complex real-life problems. However, solving LP models that have numerous decision variables and constraints requires a solution procedure such as the simplex algorithm.

The simplex algorithm is embedded in Excel's Solver add-in. We describe how LP models can be set up on Excel and solved using Solver. The structured approach presented in this chapter for setting up and solving LP problems with just two variables can be easily adapted to problems of larger size. We address several such problems in Chapter 3.

[7] For details, see N. Karmarkar. "A New Polynomial Time Algorithm for Linear Programming," *Combinatorica* 4, 4 (1984): 373–395; or J. N. Hooker. "Karmarkar's Linear Programming Algorithm," *Interfaces* 16, 4 (July–August 1986): 75–90.

Glossary

Alternate Optimal Solution A situation in which more than one optimal solution is possible. It arises when the angle or slope of the objective function is the same as the slope of the constraint.

Answer Report A report created by Solver when it solves an LP model. This report presents the optimal solution in a detailed manner.

Changing Variable Cells Cells that represent the decision variables in Solver.

Constraint A restriction (stated in the form of an inequality or an equation) that inhibits (or binds) the value that can be achieved by the objective function.

Constraint LHS The cell that contains the formula for the left-hand side of a constraint in Solver. There is one such cell for each constraint in a problem.

Constraint RHS The cell that contains the value (or formula) for the right-hand side of a constraint in Solver. There is one such cell for each constraint in a problem.

Corner (or Extreme) Point A point that lies on one of the corners of the feasible region. This means that it falls at the intersection of two constraint lines.

Corner Point Method The method of finding the optimal solution to an LP problem that involves testing the profit or cost level at each corner point of the feasible region. The theory of LP states that the optimal solution must lie at one of the corner points.

Decision Variables The unknown quantities in a problem for which optimal solution values are to be found.

Feasible Region The area that satisfies all of a problem's resource restrictions—that is, the region where all constraints overlap. All possible solutions to the problem lie in the feasible region.

Feasible Solution Any point that lies in the feasible region. Basically, it is any point that satisfies all of the problem's constraints.

Inequality A mathematical expression that contains a greater-than-or-equal-to relation ($\geq$) or a less-than-or-equal-to relation ($\leq$) between the left-hand side and the right-hand side of the expression.

Infeasible Solution Any point that lies outside the feasible region. It violates one or more of the stated constraints.

Infeasibility A condition that arises when there is no solution to an LP problem that satisfies all of the constraints.

Integer Programming A mathematical programming model in which some or all decision variables are restricted only to integer values.

Iterative Process A process (algorithm) that repeats the same steps over and over.

Level (or Iso) Line A straight line that represents all nonnegative combinations of the decision variables for a particular profit (or cost) level.

Linear Programming (LP) A mathematical technique used to help management decide how to make the most effective use of an organization's resources.

Make Unconstrained Variables Non-Negative An option available in Solver that automatically enforces the nonnegativity constraint.

Mathematical Programming The general category of mathematical modeling and solution techniques used to allocate resources while optimizing a measurable goal; LP is one type of programming model.

Nonnegativity Constraints A set of constraints that requires each decision variable to be nonnegative; that is, each decision variable must be greater than or equal to 0.

Objective Cell The cell that contains the formula for the objective function in Solver.

Objective Function A mathematical statement of the goal of an organization, stated as an intent to maximize or minimize some important quantity, such as profit or cost.

Product Mix Problem A common LP problem that involves a decision about which products a firm should produce, given that it faces limited resources.

Redundant Constraint A constraint that does not affect the feasible solution region.

Simplex Method An iterative procedure for solving LP problems.

Simplex LP An option available in Solver that forces it to solve the model as a linear program by using the simplex procedure.

Simultaneous Equation Method The algebraic means of solving for the intersection point of two or more linear constraint equations.

Slack The difference between the right-hand side and left-hand side of a $\leq$ constraint. Slack typically represents the unused resource.

Solver An Excel add-in that allows LP problems to be set up and solved in Excel.

SUMPRODUCT An Excel function that allows users to easily model formulas for the objective function and constraints while setting up a linear programming model in Excel.

Surplus The difference between the left-hand side and right-hand side of a $\geq$ constraint. Surplus typically represents the level of oversatisfaction of a requirement.

Unbounded Solution A condition that exists when the objective value can be made infinitely large (in a maximization problem) or small (in a minimization problem) without violating any of the problem's constraints.

Solved Problems

Solved Problem 2-1

Solve the following LP model graphically and then by using Excel:

$$\text{Maximize profit} = \$30X + \$40Y$$

subject to the constraints

$$\begin{aligned} 4X + 2Y &\le 16 \\ Y &\le 2 \\ 2X - Y &\ge 2 \\ X, Y &\ge 0 \end{aligned}$$

Solution

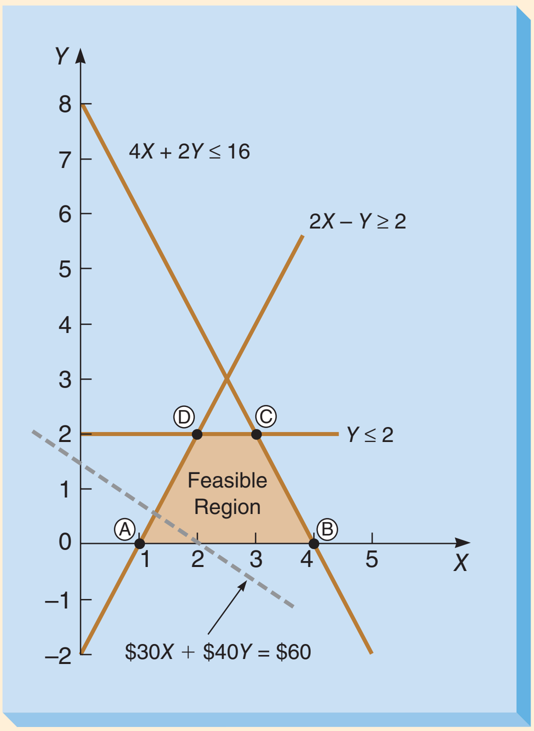
Figure 2.14 shows the feasible region as well as a level profit line for a profit value of \$60. Note that the third constraint ($2X - Y \ge 2$) has a positive slope. As usual, to find the optimal corner point, we need to move the level profit line in the direction of increased profit—that is, up and to the right. Doing so indicates that corner point C yields the highest profit. The values at this point are calculated to be $X = 3$ and $Y = 2$, yielding an optimal profit of \$170.

WWW

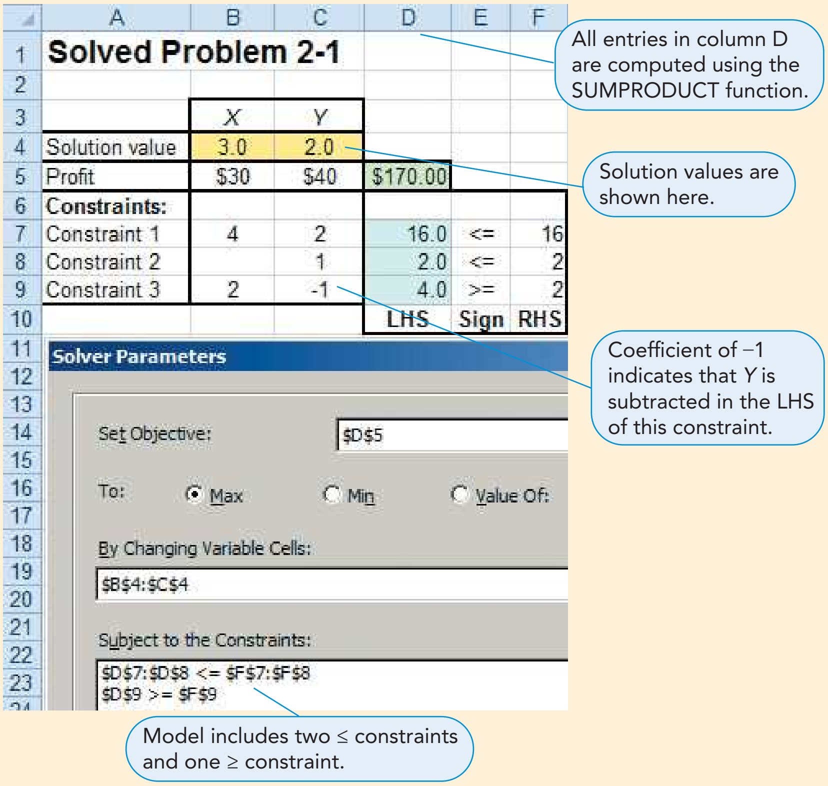
File: 2-4.xls

The Excel layout and Solver entries for this problem are shown in Screenshot 2-4. As expected, the optimal solution is the same as the one we found by using the graphical approach ($X = 3$, $Y = 2$, profit $= \$170$).

FIGURE 2.14
Graph for Solved Problem 2-1



SCREENSHOT 2-4
Excel Layout and Solver Entries for Solved Problem 2-1



Solved Problem 2-2

Solve the following LP formulation graphically and then by using Excel:

$$\text{Minimize cost} = \$24X + \$28Y$$

subject to the constraints

$$\begin{aligned} 5X + 4Y &\le 2{,}000 \\ X + Y &\ge 300 \\ X &\ge 80 \\ Y &\ge 100 \\ X, Y &\ge 0 \end{aligned}$$

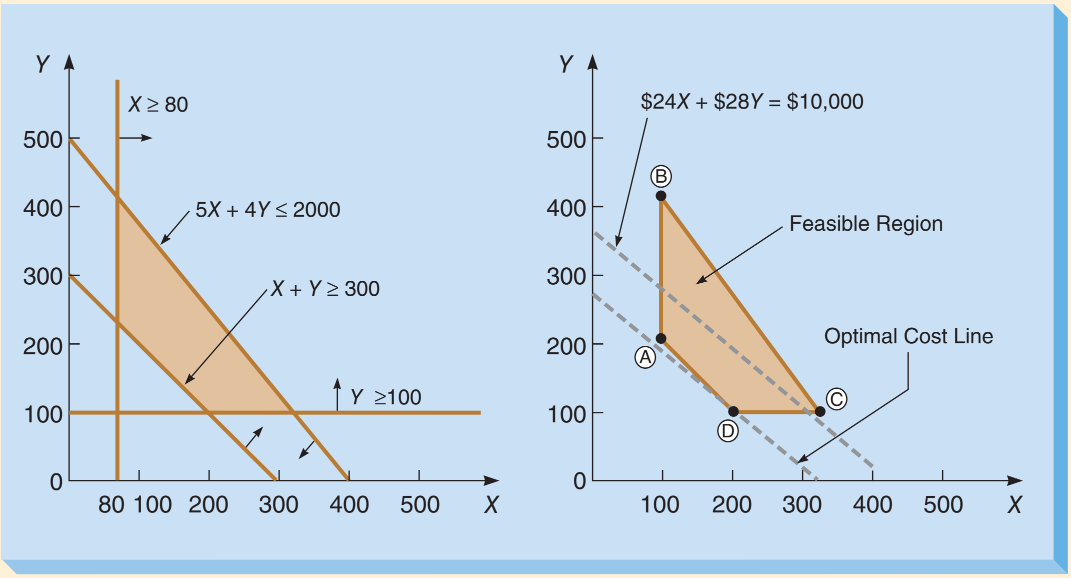
Solution

Figure 2.15 shows a graph of the feasible region along with a level line for a cost value of \$10,000. The arrows on the constraints indicate the direction of feasibility for each constraint. To find the optimal corner point, we need to move the cost line in the direction of lower cost—that is, down and to the left. The last point where a level cost line touches the feasible region as it moves toward the origin is corner point Ⓓ. Thus Ⓓ, which represents $X = 200$, $Y = 100$, and a cost of \$7,600, is the optimal solution.

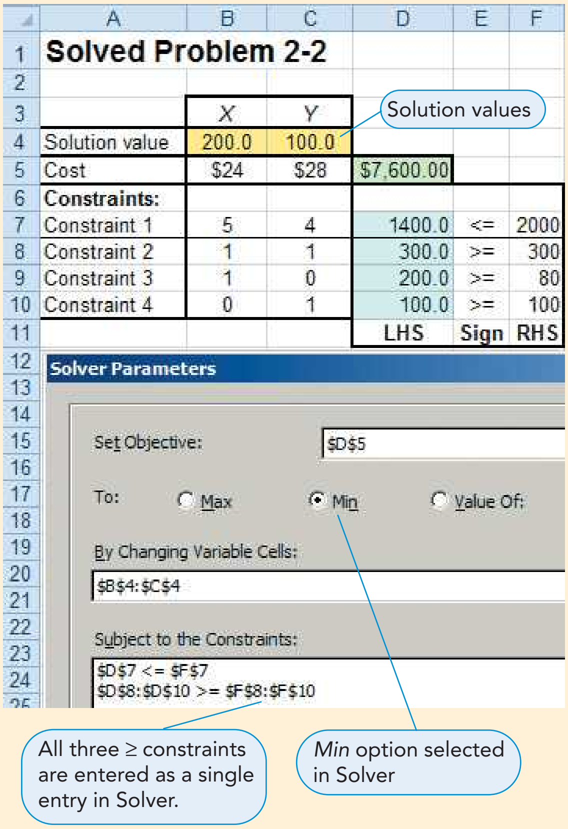
The Excel layout and Solver entries for this problem are shown in Screenshot 2-5. As expected, we get the same optimal solution as we do by using the graphical approach ($X = 200$, $Y = 100$, cost $= \$7{,}600$).

WWW
File: 2-5.xls

FIGURE 2.15
Graphs for Solved Problem 2-2



SCREENSHOT 2-5
Excel Layout and Solver Entries for Solved Problem 2-2



Discussion Questions and Problems

Discussion Questions

2-1 It is important to understand the assumptions underlying the use of any quantitative analysis model. What are the assumptions and requirements for an LP model to be formulated and used?

2-2 It has been said that each LP problem that has a feasible region has an infinite number of solutions. Explain.

2-3 Under what condition is it possible for an LP problem to have more than one optimal solution?

2-4 Under what condition is it possible for an LP problem to have an unbounded solution?

2-5 Develop your own set of constraint equations and inequalities and use them to illustrate graphically each of the following conditions:
(a) An unbounded problem
(b) An infeasible problem
(c) A problem containing redundant constraints

2-6 The production manager of a large Cincinnati manufacturing firm once made the statement, "I would like to use LP, but it's a technique that operates under conditions of certainty. My plant doesn't have that certainty; it's a world of uncertainty. So LP can't be used here." Do you think this statement has any merit? Explain why the manager may have said it.

2-7 The mathematical relationships that follow were formulated by an operations research analyst at the Smith-Lawton Chemical Company. Which ones are invalid for use in an LP problem? Why?

$$\text{Maximize profit} = 4X_1 + 3X_1X_2 + 8X_2 + 5X_3$$

subject to the constraints

$$\begin{aligned} 2X_1 + X_2 + 2X_3 &\le 50 \\ X_1 - 4X_2 &\ge 6 \\ 1.5X_1^2 + 6X_2 + 3X_3 &\ge 21 \\ 19X_2 - 0.33X_3 &= 17 \\ 5X_1 + 4X_2 + 3\sqrt{X_3} &\le 80 \end{aligned}$$

2-8 How do computers aid in solving LP problems today?

2-9 Explain why knowing how to use Excel to set up and solve LP problems may be beneficial to a manager.

2-10 What are the components of defining a problem in Excel so that it can be solved using Solver?

2-11 How is the slack (or surplus) calculated for a constraint? How is it interpreted?

2-12 What is an unbounded solution? How does Solver indicate that a problem solution is unbounded?

Problems

2-13 Solve the following LP problem by using the graphical procedure and by using Excel:

$$\text{Maximize profit} = 2X + Y$$

subject to the constraints

$$\begin{aligned} 3X + 6Y &\le 32 \\ 7X + Y &\le 20 \\ 3X - Y &\ge 3 \\ X, Y &\ge 0 \end{aligned}$$

2-14 Solve the following LP problem by using the graphical procedure and by using Excel:

$$\text{Maximize profit} = 4X + 5Y$$

subject to the constraints

$$\begin{aligned} 5X + 2Y &\le 40 \\ 3X + 6Y &\le 30 \\ X &\le 7 \\ 2X - Y &\ge 3 \\ X, Y &\ge 0 \end{aligned}$$

2-15 Solve the following LP problem by using the graphical procedure and by using Excel:

$$\text{Maximize profit} = 4X + 3Y$$

subject to the constraints

$$\begin{aligned} 2X + 4Y &\le 72 \\ 3X + 6Y &\ge 27 \\ -3X + 10Y &\ge 0 \\ X, Y &\ge 0 \end{aligned}$$

2-16 Solve the following LP problem by using the graphical procedure and by using Excel:

$$\text{Minimize cost} = 4X + 7Y$$

subject to the constraints

$$\begin{aligned} 2X + 3Y &\ge 60 \\ 4X + 2Y &\ge 80 \\ X &\le 24 \\ X, Y &\ge 0 \end{aligned}$$

2-17 Solve the following LP problem by using the graphical procedure and by using Excel:

$$\text{Minimize cost} = 3X + 7Y$$

subject to the constraints

$$\begin{aligned} 9X + 3Y &\ge 36 \\ 4X + 5Y &\ge 40 \\ X - Y &\le 0 \\ 2X &\le 6 \\ X, Y &\ge 0 \end{aligned}$$

2-18 Solve the following LP problem by using the graphical procedure and by using Excel:

$$\text{Minimize cost} = 4X + 7Y$$

subject to the constraints

$$\begin{aligned} 3X + 6Y &\ge 100 \\ 10X + 2Y &\ge 160 \\ 2Y &\ge 40 \\ 2X &\le 75 \\ X, Y &\ge 0 \end{aligned}$$

2-19 Solve the following LP problem, which involves three decision variables, by using Excel:

$$\text{Maximize profit} = 20A + 25B + 30C$$

subject to the constraints

$$\begin{aligned} 10A + 15B - 8C &\le 45 \\ 0.5(A + B + C) &\le A \\ A &\le 3B \\ B &\ge C \\ A, B, C &\ge 0 \end{aligned}$$

2-20 Consider the following four LP formulations. Using a graphical approach in each case, determine
(a) which formulation has more than one optimal solution.
(b) which formulation has an unbounded solution.
(c) which formulation is infeasible.
(d) which formulation has a unique optimal solution.

Formulation 1

maximize: $3X + 7Y$

subject to:
$$\begin{aligned} 2X + Y &\le 6 \\ 4X + 5Y &\le 20 \\ 2Y &\le 7 \\ 2X &\ge 7 \\ X, Y &\ge 0 \end{aligned}$$

Formulation 2

maximize: $3X + 6Y$

subject to:
$$\begin{aligned} 7X + 6Y &\le 42 \\ X + 2Y &\le 10 \\ X &\le 4 \\ 2Y &\le 9 \\ X, Y &\ge 0 \end{aligned}$$

Formulation 3

maximize: $2X + 3Y$

subject to:
$$\begin{aligned} X + 2Y &\ge 12 \\ 8X + 7Y &\ge 56 \\ 2Y &\ge 5 \\ X &\le 9 \\ X, Y &\ge 0 \end{aligned}$$

Formulation 4

maximize: $3X + 4Y$

subject to:
$$\begin{aligned} 3X + 7Y &\le 21 \\ 2X + Y &\le 6 \\ X + Y &\ge 2 \\ 2X &\ge 2 \\ X, Y &\ge 0 \end{aligned}$$

Note: ***Problems 2-21 to 2-36 each involve only two decision variables. Therefore, at the discretion of the instructor, they can be solved using the graphical method, Excel, or both.***

2-21 A decorating store specializing in do-it-your-self home decorators must decide how many information packets to prepare for the summer decorating season. The store managers know they will require at least 400 copies of their popular painting packet. They believe their new information packet on specialty glazing techniques could be a big seller, so they want to prepare at least 300 copies. Their printer has given the following information: The painting packet will require 2.5 minutes of printing time and 1.8 minutes of collating time. The glazing packet will require 2 minutes for each operation. The store has decided to sell the painting packet for \$5.50 a copy and to price the glazing packet at \$4.50. At this time, the printer can devote 36 hours to printing and 30 hours to collation. He will charge the store \$1 for each packet prepared. How many of each packet should the store order to maximize the revenue associated with information packets, and what is the store's expected revenue?

2-22 The Coastal Tea Company sells 60-pound bags of blended tea to restaurants. To be able to label the tea as South Carolina Tea, at least 55% of the tea (by weight) in the bag must be Carolina grown. For quality, Coastal requires that the blend achieve an average aroma rating of at least 1.65. Carolina tea, which costs Coastal \$1.80 per pound, has an aroma rating of 2; other teas likely to be blended with Carolina tea are only rated at 1.2, but they are available for only \$0.60 per pound. Determine the best mix of Carolina and regular tea to achieve Coastal's blending goals, while keeping the costs as low as possible.

2-23 The advertising agency promoting a new product is hoping to get the best possible exposure in terms of the number of people the advertising reaches. The agency will use a two-pronged approach: focused Internet advertising, which is estimated to reach 200,000 people for each burst of advertising, and print media, which is estimated to reach 80,000 people each time an ad is placed. The cost of each Internet burst is \$3,000, as opposed to only \$900 for each print media ad. It has been agreed that the number of print media ads will be no more than five times the number of Internet bursts. The agency hopes to launch at least 5 and no more than 15 Internet bursts of advertising. The advertising budget is \$75,000. Given these constraints, what is the most effective advertising strategy?

2-24 A small motor manufacturer makes two types of motor, models A and B. The assembly process for each is similar in that both require a certain amount of wiring, drilling, and assembly. Each model A takes 3 hours of wiring, 2 hours of drilling, and 1.5 hours of assembly. Each model B must go through 2 hours of wiring, 1 hour of drilling, and 0.5 hours of assembly. During the next production period, 240 hours of wiring time, 210 hours of drilling time, and 120 hours of assembly time are available. Each model A sold yields a profit of \$22. Each model B can be sold for a \$15 profit. Assuming that all motors that are assembled can be sold, find the best combination of motors to yield the highest profit.

2-25 The manufacturer in Problem 2-24 now has a standing order of 24 model A motors for each production period. Resolve Problem 2-24 to include this additional constraint.

2-26 A furniture cabinet maker produces two types of cabinets that house and hide plasma televisions. The Mission-style cabinet requires \$340 in materials and 15 labor hours to produce, and it yields a profit of \$910 per cabinet. The Rustic-style cabinet requires \$430 in materials and 20 hours to produce, and it yields a profit of \$1,200. The firm has a budget of \$30,000 to spend on materials. To ensure full employment, the firm wishes to plan to maximize its profit but at the same time to keep all 30 workers fully employed, so all 1,200 available labor hours available must be used. What is the best combination of furniture cabinets to be made?

2-27 Members of a book club have decided, after reading an investment book, to begin investing in the stock market. They would like to achieve the following: Their investment must grow by at least \$6,000 in the long term (over three years) and at least \$900 in the short term, and they must earn a dividend of at least \$300 per year. They are consulting with a stock broker, who has narrowed their search to two stocks: Carolina Solar Power and South West Steel. The data on each stock, per \$1 invested, are as follows:

	CAROLINA SOLAR POWER	SOUTH WEST STEEL
Short-term appreciation	\$0.46	\$0.26
Long-term appreciation	\$1.72	\$1.93
Dividend income	9%	13%

Assuming that these data are indicative of what will happen to these stocks over the next three years, what is the smallest investment, in dollars, that the members would have to make in one or both of these two stocks to meet their investment goals?

2-28 Treetops Hammocks produces lightweight nylon hammocks designed for campers, scouts, and hikers. The hammocks come in two styles: double and single. The double hammocks sell for \$225 each. They incur a direct labor cost of \$101.25 and a production cost of \$38.75, and they are packed with hanging apparatus and storage bags, which cost \$20. The single hammocks sell for \$175 each. Their direct labor costs are \$70 and production costs are \$30, and they too are packed with the same hanging apparatus and storage bags, which cost \$20. Each double hammock uses 3.2 hours of production time; each single hammock uses 2.4 hours of production time. Treetops plans for no more than 960 labor hours per production cycle. Treetops wants to maximize its profit while making no more than 200 single hammocks and no more than 400 total hammocks per production cycle.

(a) How many of each hammock should Treetops make?

(b) If the restriction on single hammocks were removed, what would be the optimal production plan?

2-29 A commuter airline makes lattes in the galley and sells them to passengers. A regular latte contains a shot of espresso, 1 cup of 2% milk, frothed, and 0.5 cup of whipped cream. The low-fat latte contains a shot of espresso, 1.25 cups of skim milk, frothed, and no whipped cream. The plane begins its journey with 100 shots of espresso, 60 cups of skim milk, 60 cups of 2% milk, and 30 cups of whipped cream. The airline makes a profit of \$1.58 on each regular latte and \$1.65 on each low-fat latte. Assuming that all lattes that are made can be sold, what would be the ideal mix of regular and low-fat lattes to maximize the profit for the airline?

2-30 A warehouse storage building company must determine how many storage sheds of each size—large or small—to build in its new 8,000-square-foot facility to maximize rental income. Each large shed is 150 square feet in size, requires \$1 per week in advertising, and rents for \$50 per week. Each small shed is 50 square feet in size, requires \$1 per week in advertising, and rents for \$20 per week. The company has a weekly advertising budget of \$100 and estimates that it can rent no more than 40 large sheds in any given week.

2-31 A bank is retrofitting part of its vault to hold safety deposit boxes. It plans to build safety deposit boxes approximately 6 feet high along the walls on both sides of a 20-foot corridor. Hence, the bank will have 240 square feet of wall space to use. It plans to offer two sizes of safety deposit box: large and small. Large boxes (which consume 122.4 square inches of wall space) will rent for \$40 per year. Small boxes (which consume 72 square inches of wall space) will rent for \$30 per year. The bank believes it will need at least 350 total boxes, at least 80 of which should be large. It hopes to maximize revenue for safety deposit boxes. How many boxes of each size should the bank's design provide?

2-32 An investment broker has been given \$250,000 to invest in a 12-month commitment. The money can be placed in Treasury notes (with a return of 8% and a risk score of 2) or in municipal bonds (with a return of 9% and a risk score of 3). The broker's client wants diversification to the extent that between 50% and 70% of the total investment must be placed in Treasury notes. Also, because of fear of default, the client requests that the average risk score of the total investment should be no more than 2.42. How much should the broker invest in each security so as to maximize return on investment?

2-33 A wooden furniture company manufactures two products, benches and picnic tables, for use in yards and parks. The firm has two main resources: its carpenters (labor force) and a supply of redwood for use in the furniture. During the next production cycle, 1,000 hours of labor are available. The firm also has a stock of 3,500 board-feet of good-quality redwood. Each bench that Outdoor Furniture produces requires 4 labor hours and 10 board-feet of redwood; each picnic table takes 6 labor hours and 35 board-feet of redwood. Completed benches will yield a profit of \$9 each, and tables will result in a profit of \$20 each. Since most customers usually buy tables and benches at the same time, the number of benches made should be at least twice as many as the number of tables made. How many benches and tables should be produced to obtain the largest possible profit?

2-34 A plumbing manufacturer makes two lines of bathtubs, model A and model B. Every tub requires blending a certain amount of steel and zinc; the company has available a total of 24,500 pounds of steel and 6,000 pounds of zinc. Each model A bathtub requires a mixture of 120 pounds of steel and 20 pounds of zinc, and each yields a profit of $90. Each model B tub produced can be sold for a profit of $70; it requires 100 pounds of steel and 30 pounds of zinc. To maintain an adequate supply of both models, the manufacturer would like the number of model A tubs made to be no more than 5 times the number of model B tubs. Find the best product mix of bathtubs.

2-35 A technical college department head must plan the course offerings for the next term. Student demands make it necessary to offer at least 20 core courses (each of which counts for 3 credit hours) and 20 elective courses (each of which counts for 4 credit hours) in the term. Faculty contracts dictate that a total of at least 60 core and elective courses and at least 205 total credit hours be offered. Each core course taught costs the college an average of $2,600 in faculty salaries and each elective course costs $3,000. How many each of core and elective courses should be scheduled so that total faculty salaries are kept to a minimum?

2-36 The size of the yield of olives in a vineyard is greatly influenced by a process of branch pruning. If olive trees are pruned, trees can be planted more densely, and output is increased. (However, olives from pruned trees are smaller in size.) Obtaining a barrel of olives in a pruned vineyard requires 5 hours of labor and 1 acre of land. Obtaining a barrel of olives by the normal process requires only 2 labor hours but takes 2 acres of land. A barrel of olives produced on pruned trees sell for $20, whereas a barrel of regular olives has a market price of $30. An olive grower has 250 hours of labor available and a total of 150 acres available to plant. He has determined that because of uncertain demand, no more than 40 barrels of pruned olives should be produced. Find the combination of barrels of pruned and regular olives that will yield the maximum possible profit. Also, how many acres should the olive grower devote to each growing process?

Note: ***Problems 2-37 to 2-43 are straightforward extensions of the two-variable problems we have seen so far and involve more than two variables. They therefore cannot be solved graphically. They are intended to give you an excellent opportunity to get familiar with formulating larger LP problems and solving them using Excel.***

2-37 Cattle are sent to a feedlot to be grain-fed before being processed into beef. The owners of a feedlot seek to determine the amounts of cattle feed to buy so that minimum nutritional standards are satisfied to ensure proper weight gain, while total feed costs are minimized. The feed mix used is made up of three grains that contain the following nutrients per pound of feed:

	NUTRIENT (OUNCES PER POUND OF FEED)			
FEED	**A**	**B**	**C**	**D**
Feed mix X	3	2	1	6
Feed mix Y	2	3	0	8
Feed mix Z	4	1	2	4

Feed mixes X, Y, and Z cost $3, $4, and $2.25 per pound, respectively. The minimum requirement per cattle per day is 4 pounds of nutrient A, 5 pounds of nutrient B, 1 pound of nutrient C, and 8 pounds of nutrient D. The ranch faces one additional restriction: it can only obtain 500 pounds of feed mix Z per day from the feed supplier regardless of its need. Because there are usually 100 cattle at the feed lot at any given time, this means that no more than 5 pounds of stock Z can be counted on for use in the feed of each cattle per day. Formulate this problem as a linear program and solve it by using Excel.

2-38 The production department for an aluminum valve plant is scheduling its work for next month. Each valve must go through three separate machines during the fabrication process. After fabrication, each valve is inspected by a human being, who spends 15 minutes per valve. There are 525 inspection hours available for the month. The time required (in hours) by each machine to work on each valve is shown in the following table. Also shown are the minimum number of valves that must be produced for the month and the unit profit for each valve.

PRODUCT	V231	V242	V784	V906	CAPACITY (HOURS)
Drilling	0.40	0.30	0.45	0.35	700
Milling	0.60	0.65	0.52	0.48	890
Lathe	1.20	0.60	0.5	0.70	1,200
Unit profit	$16	$12	$13	$8	
Minimum needed	200	250	600	450	

Determine the optimal production mix for the valve plant to make the best use of its profit potential.

2-39 The bank in Problem 2-31 now wants to add a mini-size box, which will rent for $17 per year and consume 43.2 square inches of wall space. The bank still wants a total of at least 350 total boxes; of these, at least 100 should be mini boxes and at least 80 should be large boxes. However, the bank wants the total area occupied by large and mini boxes to be at most 50% of the available space. How many boxes of each type should be included to maximize revenue? If all the boxes can be rented, would the bank make more money with the addition of the mini boxes?

2-40 A photocopy machine company produces three types of laser printers—the Print Jet, the Print Desk, and the Print Pro—the sale of which earn profits of $60, $90, and $73, respectively. The Print Jet requires 2.9 hours of assembly time and 1.4 hours of testing time. The Print Desk requires 3.7 hours of assembly time and 2.1 hours of testing time. The Print Pro requires 3 hours of assembly time and 1.7 hours of testing time. The company wants to ensure that Print Desk constitutes at least 15% of the total production and Print Jet and Print Desk together constitute at least 40% of the total production. There are 3,600 hours of assembly time and 2,000 hours of testing time available for the month. What combination of printers should be produced to maximize profits?

2-41 An electronics corporation manufactures four highly technical products that it supplies to aerospace firms. Each of the products must pass through the following departments before being shipped: wiring, drilling, assembly, and inspection. The time requirement (in hours) for each unit produced, the available time in each department each month, minimum production levels for each product, and unit profits for each product are summarized in the following table:

PRODUCT	EC221	EC496	NC455	NC791	CAPACITY (HOURS)
Wiring	0.5	1.5	1.5	1.0	15,000
Drilling	0.3	1.0	2.0	3.0	17,000
Assembly	0.2	4.0	1.0	2.0	10,000
Inspection	0.5	1.0	0.5	0.5	12,000
Unit profit	$9	$12	$15	$11	
Minimum needed	150	100	300	400	

Formulate this problem and solve it by using Excel. Your solution should honor all constraints and maximize the profit.

2-42 A snack company packages and sells three different 1-pound canned party mixes: Plain Nuts, Mixed Nuts, and Premium Mix. Plain Nuts sell for $2.25 per can, Mixed Nuts sell for $3.37, and Premium Nuts sell for $6.49 per can. A can of Plain Nuts contains 0.8 pound of peanuts and 0.2 pound of cashews. A can of Mixed Nuts consists of 0.5 pound of peanuts, 0.3 pound of cashews, 0.1 pound of almonds, and 0.1 pound of walnuts. A can of Premium Nuts is made up of 0.3 pound of cashews, 0.4 pound of almonds, and 0.4 pound of walnuts. The company has on hand 500 pounds of peanuts, 225 pounds of cashews, 100 pounds of almonds, and 80 pounds of walnuts. Past demand indicates that customers purchase at least twice as many cans of Plain Nuts as Premium Nuts. What production plan will maximize the total revenue?

2-43 An investor is considering three different television news stocks to complement his portfolio: British Broadcasting Company (BBC), Canadian Broadcasting Company (CBC), and Australian Broadcasting Company (ABC). His broker has given him the following information:

	SHORT-TERM GROWTH (PER $ INVESTED)	INTERMEDIATE GROWTH (PER $ INVESTED)	DIVIDEND RATE
BBC	0.39	1.59	8%
CBC	0.26	1.70	4%
ABC	0.42	1.45	6%

The investor's criteria are as follows: (1) The investment should yield short-term growth of at least $1,000; (2) the investment should yield intermediate-term growth of at least $6,000; and (3) the dividends should be at least $250 per year. Determine the least amount the investor can invest and how that investment should be allocated between the three stocks.

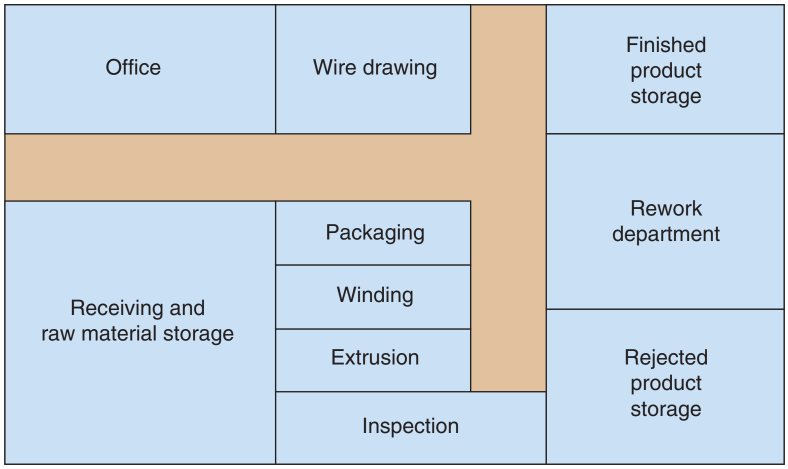
Case Study

Mexicana Wire Winding, Inc.

Ron Garcia felt good about his first week as a management trainee at Mexicana Wire Winding, Inc. He had not yet developed any technical knowledge about the manufacturing process, but he had toured the entire facility, located in the suburbs of Mexico City, and had met many people in various areas of the operation.

Mexicana, a subsidiary of Westover Wire Works, a Texas firm, is a medium-sized producer of wire windings used in making electrical transformers. Carlos Alverez, the production control manager, described the windings to Garcia as being of standardized design. Garcia's tour of the plant, laid out by process type (see Figure 2.16), followed the manufacturing sequence for the windings: drawing, extrusion, winding, inspection, and packaging. After inspection, good product is packaged and sent to finished product storage; defective product is stored separately until it can be reworked.

FIGURE 2.16
Mexicana Wire Winding Inc.



On March 8, Vivian Espania, Mexicana's general manager, stopped by Garcia's office and asked him to attend a staff meeting at 1:00 P.M.

"Let's get started with the business at hand," Vivian said, opening the meeting. "You all have met Ron Garcia, our new management trainee. Ron studied operations management in his MBA program in southern California, so I think he is competent to help us with a problem we have been discussing for a long time without resolution. I'm sure that each of you on my staff will give Ron your full cooperation."

Vivian turned to José Arroyo, production control manager, "José, why don't you describe the problem we are facing?"

"Well," José said, "business is very good right now. We are booking more orders than we can fill. We will have some new equipment on line within the next several months, which will take care of our capacity problems, but that won't help us in April. I have located some retired employees who used to work in the drawing department, and I am planning to bring them in as temporary employees in April to increase capacity there. Because we are planning to refinance some of our long-term debt, Vivian wants our profits to look as good as possible in April. I'm having a hard time figuring out which orders to run and which to back-order so that I can make the bottom line look as good as possible. Can you help me with this?"

Garcia was surprised and apprehensive to receive such an important, high-profile assignment so early in his career. Recovering quickly, he said, "Give me your data and let me work with them for a day or two."

April Orders

Product W0075C	1,400 units
Product W0033C	250 units
Product W0005X	1,510 units
Product W0007X	1,116 units

Note: Vivian Espania has given her word to a key customer that Mexicana will manufacture 600 units of product W0007X and 150 units of product W0075C for him during April.

Standard Cost

PRODUCT	MATERIAL	LABOR	OVERHEAD	SELLING PRICE
W0075C	$33.00	$9.90	$23.10	$100.00
W00033C	25.00	7.50	17.50	80.00
W0005X	35.00	10.50	24.50	130.00
W0007X	75.00	11.25	63.75	175.00

Selecting Operating Data

Average output per month = 2,400 units

Average machine utilization = 63%

Average percentage of production sent to rework department = 5% (mostly from winding department)

Average no. of rejected units awaiting rework = 850 (mostly from winding department)

Plant Capacity (HOURS)

DRAWING	EXTRUSION	WINDING	PACKAGING
4,000	4,200	2,000	2,300

Note: Inspection capacity is not a problem: Employees can work overtime as necessary to accommodate any schedule.

Bill of Labor (HOURS/UNIT)

PRODUCT	DRAWING	EXTRUSION	WINDING	PACKAGING
W0075C	1.0	1.0	1.0	1.0
W0033C	2.0	1.0	3.0	0.0
W0005X	0.0	4.0	0.0	3.0
W0007X	1.0	1.0	0.0	2.0

Discussion Questions

1. What recommendations should Ron Garcia make, with what justification? Provide a detailed analysis, with charts, graphs, and Excel printouts included.
2. Discuss the need for temporary workers in the drawing department.
3. Discuss the plant layout.

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Case Study

Golding Landscaping and Plants, Inc.

Kenneth and Patricia Golding spent a career as a husband-and-wife real estate investment partnership in Washington, DC. When they finally retired to a 25-acre farm in northern Virginia's Fairfax County, they became ardent amateur gardeners. Kenneth planted shrubs and fruit trees, and Patricia spent her hours potting all sizes of plants. When the volume of shrubs and plants reached the point that the Goldings began to think of their hobby in a serious vein, they built a greenhouse adjacent to their home and installed heating and watering systems.

By 2005, the Goldings realized that their retirement from real estate had really only led to a second career—in the plant and shrub business—and they filed for a Virginia business license. Within a matter of months, they asked their attorney to file incorporation documents and formed the firm Golding Landscaping and Plants, Inc.

Early in the new business's existence, Kenneth Golding recognized the need for a high-quality commercial fertilizer that he could blend himself, both for sale and for his own nursery. His goal was to keep his costs to a minimum while producing a top-notch product that was especially suited to the northern Virginia climate.

Working with chemists at George Mason University, Golding blended "Golding-Grow." It consists of four chemical compounds: C-30, C-92, D-21, and E-11. The cost per pound for each compound is indicated in the following table:

CHEMICAL COMPOUND	COST PER POUND ($)
C-30	0.12
C-92	0.09
D-21	0.11
E-11	0.04

The specifications for Golding-Grow are as follows:

a. Chemical E-11 must comprise at least 15% of the blend.
b. C-92 and C-30 must together constitute at least 45% of the blend.
c. D-21 and C-92 can together constitute no more than 30% of the blend.
d. Golding-Grow is packaged and sold in 50-pound bags.

Discussion Questions

1. Formulate an LP problem to determine what blend of the four chemicals will allow Golding to minimize the cost of a 50-pound bag of the fertilizer.
2. Solve by using Excel to find the best solution.

Source: J. Heizer and B. Render. *Operations Management* Eighth Edition, p. 720, © 2006. Reprinted by permission of Pearson Education, Inc., Upper Saddle River, NJ.

Internet Case Studies

See the Companion Website for this textbook, at www.pearsonhighered.com/balakrishnan, for additional case studies.

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CHAPTER 3

Linear Programming Modeling Applications with Computer Analyses in Excel

LEARNING OBJECTIVES

After completing this chapter, students will be able to:

1. Model a wide variety of linear programming (LP) problems.
2. Understand major business application areas for LP problems, including manufacturing, marketing, finance, employee staffing, transportation, blending, and multiperiod planning.
3. Gain experience in setting up and solving LP problems using Excel's Solver.

CHAPTER OUTLINE

3.1 Introduction

The purpose of this chapter is to illustrate how linear programming (LP) can be used to model real-world problems in several managerial decision-making areas. In our discussion, we use examples from areas such as product mix, make–buy decisions, media selection, marketing research, financial portfolio selection, labor planning, shipping and transportation, allocation decisions, ingredient blending, and multiperiod scheduling.

It is a good idea to always develop a written LP model on paper before attempting to implement it on Excel.

For each example discussed, we first briefly describe the development of the written mathematical model and then illustrate its solution using Excel's Solver. Although we use Solver to solve these models, it is critical that you understand the logic behind a model before implementing it on the computer. Remember that the solution is only as good as the model itself. If the model is incorrect or incomplete from a logical perspective (even if it is correct from a mathematical perspective), Excel has no way of recognizing the logical error. Too many students, especially those at the early stages of instruction in LP, hit roadblocks when they try to implement an LP problem directly in Excel without conceptualizing the model on paper first. So we highly recommend that, until you become very comfortable with LP formulations (which takes many hours of practice), you sketch out the layout for each problem on paper first. Then, you can translate your written model to the computer.

We first identify decision variables and then write linear equations for the objective function and each of the constraints.

In developing each written mathematical model, we use the approach discussed in Chapter 2. This means first identifying the decision variables and then writing out linear equations for the objective function and each constraint in terms of these decision variables. Although some of the models discussed in this chapter are relatively small numerically, the principles developed here are definitely applicable to larger problems. Moreover, the structured formulation approach used here should provide enough practice in "paraphrasing" LP model formulations and help in developing skills to apply the technique to other, less common applications.

We use a consistent layout in our Excel implementation of all models for ease of understanding.

When implementing these models in Excel, to the extent possible, we employ the same layout presented in Chapter 2. That is, all parameters (i.e., the solution value, objective coefficient, and constraint coefficients) associated with a specific decision variable are modeled in the same column. The objective function and each constraint in the problem are shown on separate rows of the worksheet. Later in this chapter (section 3.8), however, we illustrate an alternate implementation that may be more compact and efficient for some problems. As noted in Chapter 2, we encourage you to try alternate layouts based on your personal preference and expertise with Excel.

Excel Notes

- The Companion Website for this textbook, at www.pearsonhighered.com/balakrishnan, contains the Excel file for each sample problem discussed here. The relevant file name is shown in the margin next to each example.
- In each of our Excel layouts, for clarity, changing variable cells are shaded yellow, the objective cell is shaded green, and cells denoting left-hand-side formulas of constraints are shaded blue. If the right-hand side of a constraint also includes a formula, that cell is also shaded blue.
- To make the equivalence of the written formulation and the Excel layout clear, the Excel layouts show the decision variable names used in the written formulation of the model. Note that these names have no role in using Solver to solve the model.

3.2 Manufacturing Applications

Product Mix Problem

A popular use of LP is in solving product mix problems.

A fertile field for the use of LP is in planning for the optimal mix of products that a company should produce. A company must meet a myriad of constraints, ranging from financial concerns to sales demands to material contracts to union labor demands. Its primary goal is either to generate the largest profit (or revenue) possible or to keep the total manufacturing costs as low as possible. We have already studied a simple version of a product mix problem (the Flair

IN ACTION Improved Handling of Time-Sensitive Returns at Hewlett-Packard Using Linear Programming

Reverse supply chain operations at Hewlett-Packard Company (HP) include organizing product returns, identifying their best reuse option and reconditioning them accordingly, and finally marketing and selling the reconditioned products. HP estimates the cost of product returns at 2 percent of total outbound sales for North America. Managers are therefore increasingly aware of the value of product returns, especially for products that lose their value rapidly with time. Products such as PCs, printers, computer peripherals, and mobile phones have very short life cycles and in order to convert these returns to potential sources of revenue, the time between return and resale must be short.

Focusing on HP's notebooks and desktops because they constituted about 60 percent of the return flows and contributed more than 80 percent of revenue, the authors developed innovative linear programming models to explore alternative refurbishment options and improve reuse and reconditioning decisions. A spokesperson at HP notes that the "result of this project was really eye opening for my organization and delivered tremendous value towards improving our existing business model." He adds, "the results are applicable to a wide range of product returns for HP and we are exploring how to extend the results from this pilot study to other HP groups for other product lines and other regions."

Source: Based on V. D. R. Guide, Jr., L. Muyldermans, and L. N. Van Wassenhove. "Hewlett-Packard Company Unlocks the Value Potential from Time-Sensitive Returns," *Interfaces* 35, 4 (July–August 2005): 281–293.

Furniture problem) involving just two products in Chapter 2. Let us now look at a more detailed version of a product mix problem.

Fifth Avenue Industries, a nationally known manufacturer of menswear, produces four varieties of ties. One is an expensive, all-silk tie, one is an all-polyester tie, and two are blends of polyester and cotton. Table 3.1 illustrates the cost and availability (per monthly production planning period) of the three materials used in the production process.

The firm has fixed contracts with several major department store chains to supply ties each month. The contracts require that Fifth Avenue Industries supply a minimum quantity of each tie but allow for a larger demand if Fifth Avenue chooses to meet that demand. (Most of the ties are not shipped with the name Fifth Avenue on their label, incidentally, but with "private stock" labels supplied by the stores.) Table 3.2 summarizes the contract demand for each of the four styles of ties, the selling price per tie, and the fabric requirements of each variety. The production process for all ties is almost fully automated, and Fifth Avenue uses a standard labor cost of \$0.75 per tie (for any variety). Fifth Avenue must decide on a policy for product mix in order to maximize its monthly profit.

TABLE 3.1
Material Data for Fifth Avenue Industries

MATERIAL	COST PER YARD	MATERIAL AVAILABLE PER MONTH (YARDS)
Silk	\$20	1,000
Polyester	\$ 6	2,000
Cotton	\$ 9	1,250

TABLE 3.2 Product Data for Fifth Avenue Industries

VARIETY OF TIE	SELLING PRICE PER TIE	MONTHLY CONTRACT MINIMUM	MONTHLY DEMAND	TOTAL MATERIAL REQUIRED PER TIE (YARDS)	MATERIAL REQUIREMENTS
All silk	\$6.70	6,000	7,000	0.125	100% silk
All polyester	\$3.55	10,000	14,000	0.08	100% polyester
Poly–cotton blend 1	\$4.31	13,000	16,000	0.10	50% polyester/50% cotton
Poly–cotton blend 2	\$4.81	6,000	8,500	0.10	30% polyester/70% cotton

Decision variables in product mix problems usually represent the number of units to make of each product.

FORMULATING THE PROBLEM As is usual with product mix problems, in this case, the decision variables represent the number of units to make of each product. Let

$$\begin{aligned} S &= \text{number of all-silk ties to make per month} \\ P &= \text{number of all-polyester ties to make per month} \\ B_1 &= \text{number of poly–cotton blend 1 ties to make per month} \\ B_2 &= \text{number of poly–cotton blend 2 ties to make per month} \end{aligned}$$

Unlike the Flair Furniture example in Chapter 2, where the unit profit contribution for each product was directly given (e.g., \$7 per table and \$5 per chair), the unit profits must be first calculated in this example. We illustrate the net profit calculation for all-silk ties (S). Each all-silk tie requires 0.125 yards of silk, at a cost of \$20 per yard, resulting in a material cost of \$2.50. The selling price per all-silk tie is \$6.70, and the labor cost is \$0.75, leaving a net profit of $\$6.70 - \$2.50 - \$0.75 = \$3.45$ per tie. In a similar fashion, we can calculate the net unit profit for all-polyester ties (P) to be \$2.32, for poly–cotton blend 1 ties (B_1) to be \$2.81, and for poly–cotton blend 2 ties (B_2) to be \$3.25. Try to verify these calculations for yourself.

The objective function can now be stated as

$$\text{Maximize profit} = \$3.45S + \$2.32P + \$2.81B_1 + \$3.25B_2$$

subject to the constraints

$$\begin{array}{lcrl} 0.125S & \leq & 1{,}000 & \text{(yards of silk)} \\ 0.08P + 0.05B_1 + 0.03B_2 & \leq & 2{,}000 & \text{(yards of polyester)} \\ 0.05B_1 + 0.07B_2 & \leq & 1{,}250 & \text{(yards of cotton)} \\ S & \geq & 6{,}000 & \text{(contract minimum for all silk)} \\ S & \leq & 7{,}000 & \text{(maximum demand for all silk)} \\ P & \geq & 10{,}000 & \text{(contract minimum for all polyester)} \\ P & \leq & 14{,}000 & \text{(maximum demand for all polyester)} \\ B_1 & \geq & 13{,}000 & \text{(contract minimum for blend 1)} \\ B_1 & \leq & 16{,}000 & \text{(maximum demand for blend 1)} \\ B_2 & \geq & 6{,}000 & \text{(contract minimum for blend 2)} \\ B_2 & \leq & 8{,}500 & \text{(maximum demand for blend 2)} \\ S, P, B_1, B_2 & \geq & 0 & \text{(nonnegativity)} \end{array}$$

Instead of profit contributions, the objective function can include selling prices and cost components.

Instead of writing the objective function by using the profit coefficients directly, we can optionally choose to split the profit into its three components: a revenue component, a labor cost component, and a material cost component. For example, the objective (profit) coefficient for all-silk ties (S) is \$3.45. However, we know that the \$3.45 is obtained by subtracting the labor cost (\$0.75) and material cost (\$2.50) from the revenue (\$6.70) for S. Hence, we can rewrite the objective function as

$$\begin{aligned} \text{Maximize profit} = {} & (\$6.70S + \$3.55P + \$4.31B_1 + \$4.81B_2) \\ & - \$0.75(S + P + B_1 + B_2) - (\$2.50S + \$0.48P + \$0.75B_1 + \$0.81B_2) \end{aligned}$$

It is preferable to have the Excel layout show as much detail as possible for a problem.

Whether we model the objective function by using the profit coefficients directly or by using the selling prices and cost coefficients, the final solution will be the same. However, in many problems it is convenient, and probably preferable, to have the model show as much detail as possible.

WWW

File: 3-1.xls, sheet: 3-1A

SOLVING THE PROBLEM The formula view of the Excel layout for this problem is shown in Screenshot 3-1A. Cell F6 defines the revenue component, cell F7 defines the labor cost component, and cell F8 defines the material cost component of the objective function. Cell F9 (the objective cell in Solver) is the difference between cell F6 and cells F7 and F8.

Most formulas in our Excel layout are modeled using the SUMPRODUCT function.

Observe that in this spreadsheet (as well as in all other spreadsheets discussed in this chapter), the primary Excel function we have used to model all formulas is the SUMPRODUCT function (discussed in section 2.7, on page 42). We have used this function to compute the objective function components (cells F6:F8) as well as the LHS values for all constraints (cells F11:F21).

SCREENSHOT 3-1A Formula View of the Excel Layout for Fifth Avenue Industries

Titles, such as the ones shown in row 4 and column A, are included to clarify the model. They are recommended but not required.

These are the decision variable names used in the written LP formulation. They are shown here for information purposes only.

	A	B	C	D	E	F	G	H	I
1	**Fifth Avenue Industries**								
2									
3		S	P	B_1	B_2				
4		All silk	All poly	Blend-1	Blend-2				
5	Number of units								
6	Selling price	6.7	3.55	4.31	4.81	=SUMPRODUCT(B6:E6,$B$5:$E$5)			
7	Labor cost	0.75	0.75	0.75	0.75	=SUMPRODUCT(B7:E7,$B$5:$E$5)			
8	Material cost	2.5	0.48	0.75	0.81	=SUMPRODUCT(B8:E8,$B$5:$E$5)			
9	Profit	=B6-B7-B8	=C6-C7-C8	=D6-D7-D8	=E6-E7-E8	=F6-F7-F8			
10	**Constraints:**								**Cost/Yd**
11	Yards of silk	0.125				=SUMPRODUCT(B11:E11,$B$5:$E$5)	<=	1000	20
12	Yards of polyester		0.08	0.05	0.03	=SUMPRODUCT(B12:E12,$B$5:$E$5)	<=	2000	6
13	Yards of cotton			0.05	0.07	=SUMPRODUCT(B13:E13,$B$5:$E$5)	<=	1250	9
14	Maximum all silk	1				=SUMPRODUCT(B14:E14,$B$5:$E$5)	<=	7000	
15	Maximum all poly		1			=SUMPRODUCT(B15:E15,$B$5:$E$5)	<=	14000	
16	Maximum blend-1			1		=SUMPRODUCT(B16:E16,$B$5:$E$5)	<=	16000	
17	Maximum blend-2				1	=SUMPRODUCT(B17:E17,$B$5:$E$5)	<=	8500	
18	Minimum all silk	1				=SUMPRODUCT(B18:E18,$B$5:$E$5)	>=	6000	
19	Minimum all poly		1			=SUMPRODUCT(B19:E19,$B$5:$E$5)	>=	10000	
20	Minimum blend-1			1		=SUMPRODUCT(B20:E20,$B$5:$E$5)	>=	13000	
21	Minimum blend-2				1	=SUMPRODUCT(B21:E21,$B$5:$E$5)	>=	6000	
22						**LHS**	**Sign**	**RHS**	

Objective function terms and constraint LHS values are computed using the SUMPRODUCT function.

The signs are shown here for information purposes only. Actual signs will be entered in Solver.

As noted earlier, we have adopted this type of Excel layout for our models in order to make it easier for the beginning student of LP to understand them. Further, an advantage of this layout is the ease with which all the formulas in the spreadsheet can be created. Observe that we have used the $ sign in cell F6 to anchor the cell references for the decision variables (i.e., $B$5:$E$5). This allows us to simply copy this formula to cells F7:F8 for the other components of the objective function, and to cells F11:F21 to create the corresponding LHS formulas for the constraints.

WWW

File: 3-1.xls, sheet: 3-1B

INTERPRETING THE RESULTS The Solver entries and optimal solution values for this model are shown in Screenshot 3-1B. As discussed in Chapter 2, before solving the LP model using Solver, we ensure that the box labeled Make Unconstrained Variables Non-Negative is checked and the Select a Solving Method box is set to Simplex LP; both are set this way by default in Solver.

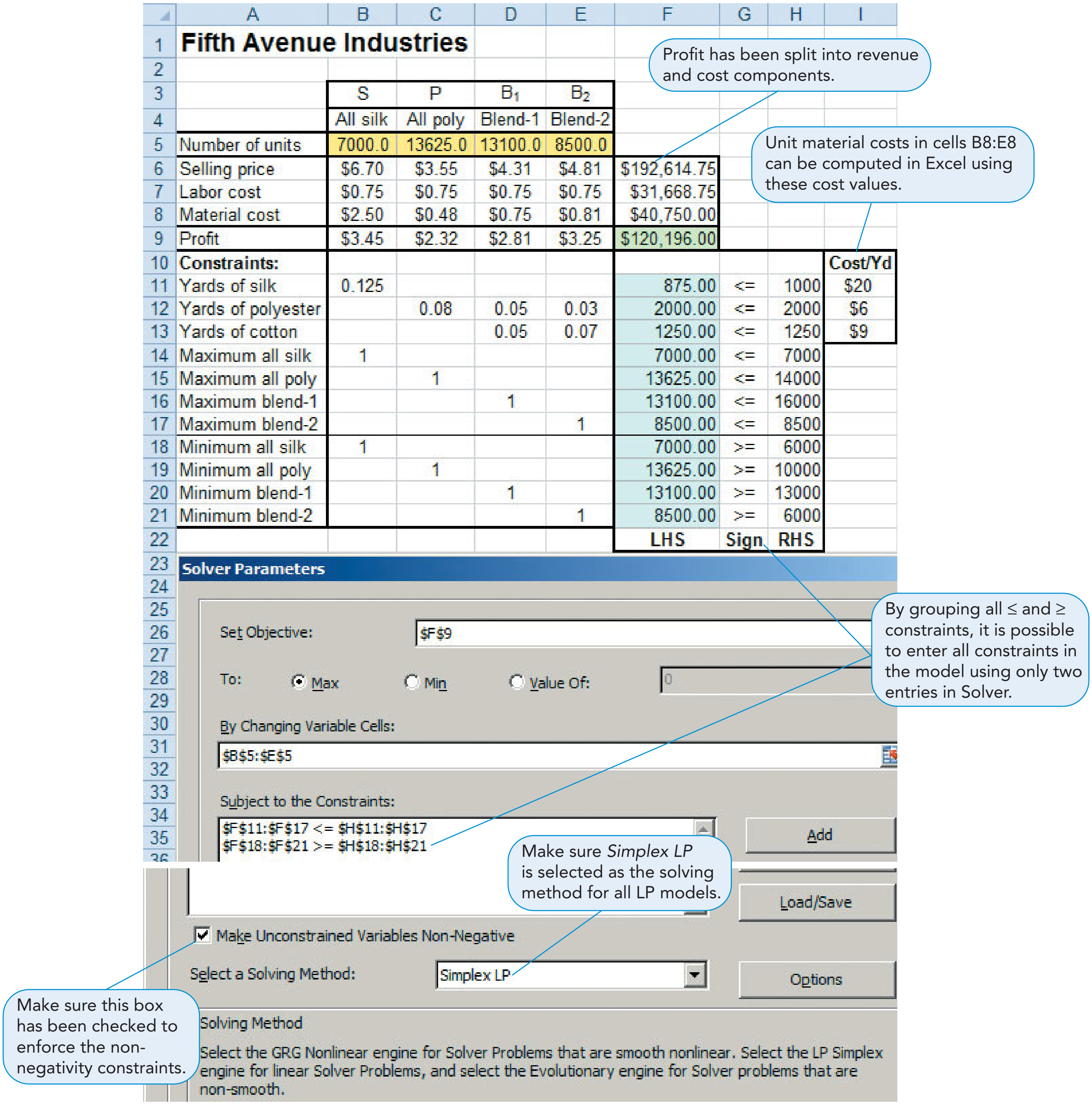
The results indicate that the optimal solution is to produce 7,000 all-silk ties, 13,625 all-polyester ties, 13,100 poly–cotton blend 1 ties, and 8,500 poly–cotton blend 2 ties. This results in total revenue of $192,614.75, labor cost of $31,668.75, and total material cost of $40,750, yielding a net profit of $120,196. Polyester and cotton availability are binding constraints, while 125 yards (= 1,000 − 875) of silk will be left unused. Interestingly, the availability of the two cheaper resources (polyester and cotton) is more critical to Fifth Avenue than the availability of the more expensive resource (silk). Such occurrences are common in practice. That is, the more expensive resources need not necessarily be the most important or critical resources from an availability or need point of view. Fifth Avenue will satisfy the full demand for all silk and poly–cotton blend 2 ties, and it will satisfy a little over the minimum contract level for the other two varieties.

Make–Buy Decision Problem

The make–buy decision problem is an extension of the product mix problem.

An extension of the product mix problem is the make–buy decision problem. In this situation, a firm can satisfy the demand for a product by making some of it in-house ("make") and by subcontracting or outsourcing the remainder to another firm ("buy"). For each product, the firm

SCREENSHOT 3-1B Excel Layout and Solver Entries for Fifth Avenue Industries



needs to determine how much of the product to make in-house and how much of it to outsource to another firm. (*Note:* Under the scenario considered here, it is possible for the firm to use *both* the in-house and outsourcing options simultaneously for a product. That is, it is not necessary for the firm to choose either the in-house option or the outsourcing option exclusively. The situation where the firm has to choose only one of the two options for a given product is an example of a binary integer programming model, which we will address in Chapter 6.)

To illustrate this type of problem, let us consider the Fifth Avenue Industries problem again. As in the product mix example, Tables 3.1 and 3.2 show the relevant data for this problem. However, let us now assume that the firm *must* satisfy all demand exactly. That is, the monthly contract minimum numbers in Table 3.2 are now the same as the monthly demands. In addition, assume that Fifth Avenue now has the option to outsource part of its tie production to Ties

Unlimited, another tie maker. Ties Unlimited has enough surplus capacity to handle any order that Fifth Avenue may place. It has provided Fifth Avenue with the following price list per tie: all silk, \$4.25; all polyester, \$2.00; poly-cotton blend 1, \$2.50; and poly-cotton blend 2, \$2.20. (Ties Unlimited is selling poly-cotton blend 2 for a lower price than poly-cotton blend 1 because it has obtained its cotton at a much cheaper cost than Fifth Avenue.) What should Fifth Avenue do under this revised situation to maximize its monthly profit?

FORMULATING THE PROBLEM As before, let

$$\begin{aligned}
S &= \text{number of all-silk ties to make (in-house) per month}\\
P &= \text{number of all-polyester ties to make (in-house) per month}\\
B_1 &= \text{number of poly–cotton blend 1 ties to make (in-house) per month}\\
B_2 &= \text{number of poly–cotton blend 2 ties to make (in-house) per month}
\end{aligned}$$

In this case, however, Fifth Avenue must decide on a policy for the product mix that includes both the make and buy options. To accommodate this feature, let

$$\begin{aligned}
S_o &= \text{number of all-silk ties to outsource (buy) per month}\\
P_o &= \text{number of all-polyester ties to outsource (buy) per month}\\
B_{1o} &= \text{number of poly–cotton blend 1 ties to outsource (buy) per month}\\
B_{2o} &= \text{number of poly–cotton blend 2 ties to outsource (buy) per month}
\end{aligned}$$

Note that the total number of each variety of tie equals the sum of the number of ties made in-house and the number of ties outsourced. For example, the total number of all-silk ties equals $S + S_o$. The total revenue from all ties can now be written as

$$\text{Revenue} = \$6.70(S + S_o) + \$3.55(P + P_o) + \$4.31(B_1 + B_{1o}) + \$4.81(B_2 + B_{2o})$$

To compute the profit, we need to subtract the labor cost, material cost, and outsourcing cost from this revenue. As in the previous product mix example, the total labor and material costs may be written as

$$\begin{aligned}
\text{Labor cost} &= \$0.75(S + P + B_1 + B_2)\\
\text{Material cost} &= \$2.50S + \$0.48P + \$0.75B_1 + \$0.81B_2
\end{aligned}$$

Note that the labor and material costs are relevant only for the portion of the production that occurs in-house. The total outsourcing cost may be written as

$$\text{Outsourcing cost} = \$4.25S_o + \$2.00P_o + \$2.50B_{1o} + \$2.20B_{2o}$$

The objective function can now be stated as

The objective function now also includes the outsourcing cost.

$$\text{Maximize profit} = \text{revenue} - \text{labor cost} - \text{material cost} - \text{outsourcing cost}$$

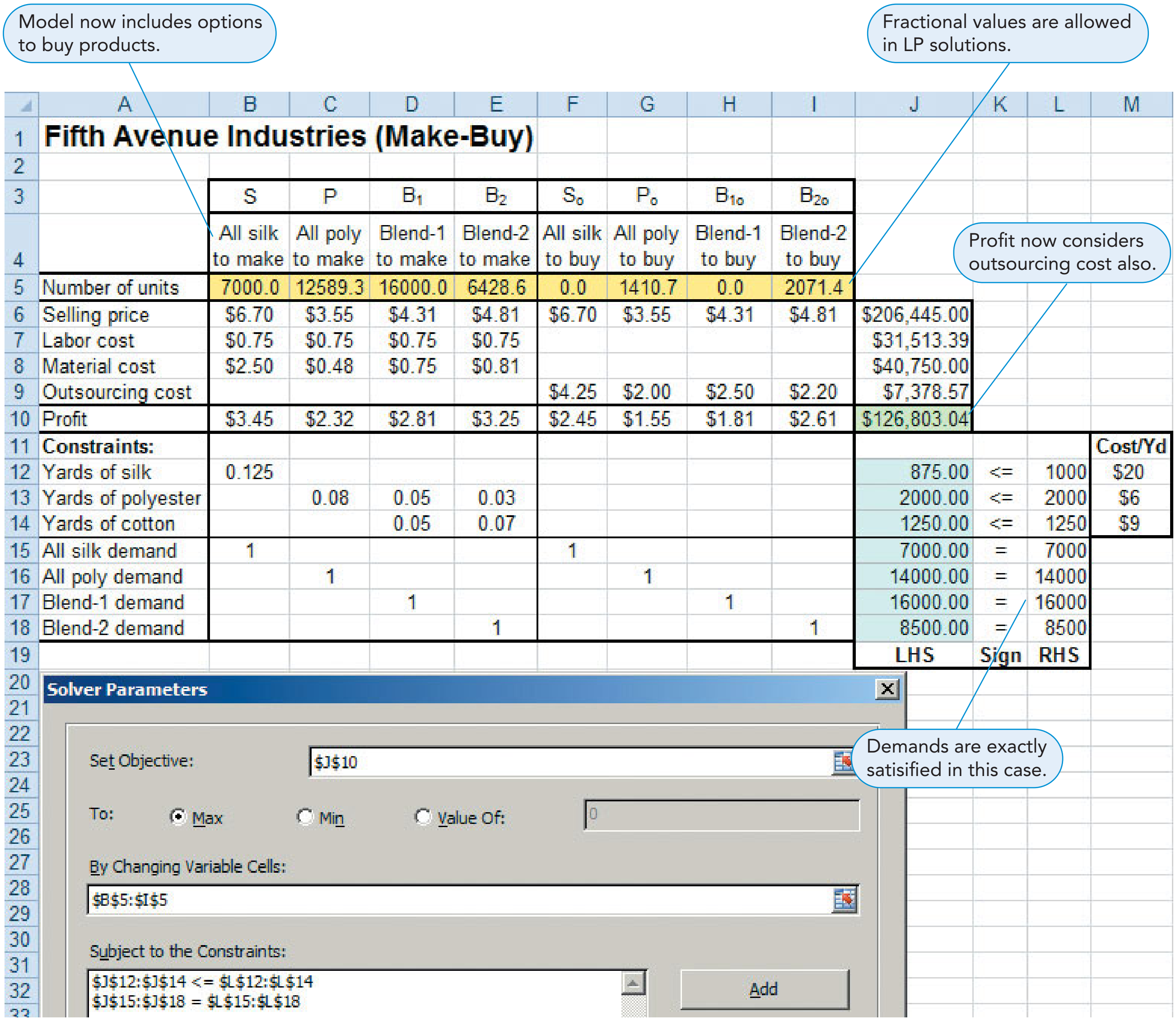
subject to the constraints

$$\begin{array}{lrrl}
0.125S & \le & 1{,}000 & \text{(yards of silk)}\\
0.08P + 0.05B_1 + 0.03B_2 & \le & 2{,}000 & \text{(yards of polyester)}\\
0.05B_1 + 0.07B_2 & \le & 1{,}250 & \text{(yards of cotton)}\\
S + S_o & = & 7{,}000 & \text{(required demand for all-silk)}\\
P + P_o & = & 14{,}000 & \text{(required demand for all-polyester)}\\
B_1 + B_{1o} & = & 16{,}000 & \text{(required demand for blend 1)}\\
B_2 + B_{2o} & = & 8{,}500 & \text{(required demand for blend 2)}\\
S, P, B_1, B_2, S_o, P_o, B_{1o}, B_{2o} & \ge & 0 & \text{(nonnegativity)}
\end{array}$$

File: 3-2.xls

Because all demand must be satisfied in this example, we write a single demand constraint for each tie variety. Note that we have written all the demand constraints as = constraints in this model. Could we have used the $\le$ sign for the demand constraints without affecting the solution? In this case, the answer is yes because we are trying to maximize profit, and the profit contribution of each tie is positive, regardless of whether it is made in-house or outsourced (as shown in cells B10:I10 in the Excel layout for this model in Screenshot 3-2).

SCREENSHOT 3-2 **Excel Layout Solver Entries for Fifth Avenue Industries—Make–Buy Problem**



Likewise, could we have used the ≥ sign for the demand constraints? In this case, the answer is no. Why? (*Hint:* Because the selling price of each tie variety is greater than its outsourcing cost, what will the solution suggest if we write the demand constraints as ≥ equations?)

SOLVING THE PROBLEM The Solver entries and optimal solution values for this model are shown in Screenshot 3-2. Cell J6 defines the revenue component of the profit, cells J7 and J8 define the labor and material cost components, respectively, and cell J9 defines the outsourcing cost component. The profit shown in cell J10 (the objective cell in Solver) is the difference between cell J6 and cells J7, J8, and J9.

INTERPRETING THE RESULTS The results show that the optimal solution is to make in-house 7,000 all-silk ties, 12,589.30 all-polyester ties, 16,000 poly–cotton blend 1 ties, and 6,428.60 poly–cotton blend 2 ties. The remaining demand (i.e., 1,410.70 all-polyester ties and 2,071.40 poly–cotton blend 2 ties) should be outsourced from Ties Unlimited. This results in total revenue of $206,445, total labor cost of $31,513.39, total material cost of $40,750, and total outsourcing cost of $7,378.57, yielding a net profit of $126,803.04. It is interesting to note that the presence of the outsourcing option changes the product mix for the in-house production (compare Screenshot 3-1B with Screenshot 3-2). Also, Fifth Avenue's total profit increases as a result of the outsourcing because it is now able to satisfy more of the demand (profit is now \$126,803.04 versus only \$120,196 earlier).

Because integer programming models are much more difficult to solve than LP models, we solve many problems as LP models and round off any fractional values.

Observe that this solution turns out to have some fractional values. As noted in Chapter 2, because it is considerably more difficult to solve an integer programming model than an LP model, it is quite common to not specify the integer requirement in many LP models. We can then round the resulting solution appropriately if it turns out to have fractional values. In product mix problems that have many $\leq$ constraints, the fractional values for production variables should typically be rounded down. If we round up these values, we might potentially violate a binding constraint for a resource. In Fifth Avenue's case, the company could probably round down P and B_2 to 12,589 and 6,428, respectively, and it could correspondingly round up the outsourced values P_o and B_{2o} to 1,411 and 2,072, respectively, without affecting the profit too much.

Rounding may be a difficult task in some situations.

In some situations, however, rounding may not be an easy task. For example, there is likely to be a huge cost and resource impact if we round a solution that suggests making 10.71 Boeing 787 aircraft to 10 versus rounding it to 11. In such cases, we need to solve the model as an integer programming problem. We discuss such problems (i.e., problems that require integer solutions) in Chapter 6.

3.3 Marketing Applications

Media Selection Problem

Media selection problems can be approached with LP from two perspectives: maximizing audience exposure or minimizing advertising costs.

LP models have been used in the advertising field as a decision aid in selecting an effective media mix. Sometimes the technique is employed in allocating a fixed or limited budget across various media, which might include radio or television commercials, newspaper ads, direct mailings, magazine ads, and so on. In other applications, the objective is to maximize audience exposure. Restrictions on the allowable media mix might arise through contract requirements, limited media availability, or company policy. An example follows.

Win Big Gambling Club promotes gambling junkets from a large Midwestern city to casinos in The Bahamas. The club has budgeted up to \$8,000 per week for local advertising. The money is to be allocated among four promotional media: TV spots, newspaper ads, and two types of radio advertisements. Win Big's goal is to reach the largest possible high-potential audience through the various media. Table 3.3 presents the number of potential gamblers reached by making use of an advertisement in each of the four media. It also provides the cost per advertisement placed and the maximum number of ads that can be purchased per week.

Win Big's contractual arrangements require that at least five radio spots be placed each week. To ensure a broad-scoped promotional campaign, management also insists that no more than \$1,800 be spent on radio advertising every week.

FORMULATING AND SOLVING THE PROBLEM This is somewhat like the product mix problem we discussed earlier in this chapter, except that the "products" here are the various media that are available for use. The decision variables denote the number of times each of these media choices should be used. Let

T = number of 1-minute television spots taken each week

N = number of full-page daily newspaper ads taken each week

P = number of 30-second prime time radio spots taken each week

A = number of 1-minute afternoon radio spots taken each week

TABLE 3.3
Data for Win Big Gambling Club

MEDIUM	AUDIENCE REACHED PER AD	COST PER AD	MAXIMUM ADS PER WEEK
TV spot (1 minute)	5,000	\$800	12
Daily newspaper (full-page ad)	8,500	\$925	5
Radio spot (30 seconds, prime time)	2,400	\$290	25
Radio spot (1 minute, afternoon)	2,800	\$380	20

Objective:

$$\text{Maximize audience coverage} = 5{,}000T + 8{,}500N + 2{,}400P + 2{,}800A$$

subject to the constraints

$$
\begin{array}{lrl}
T & \le 12 & \text{(maximum TV spots/week)} \\
N & \le 5 & \text{(maximum newspaper ads/week)} \\
P & \le 25 & \text{(maximum 30-second radio spots/week)} \\
A & \le 20 & \text{(maximum 1-minute radio spots/week)} \\
800T + 925N + 290P + 380A & \le 8{,}000 & \text{(weekly advertising budget)} \\
P + A & \ge 5 & \text{(minimum radio spots contracted)} \\
290P + 380A & \le 1{,}800 & \text{(maximum dollars spent on radio)} \\
T, N, P, A & \ge 0 & \text{(nonnegativity)}
\end{array}
$$

The Excel layout and Solver entries for this model, and the resulting solution, are shown in Screenshot 3-3.

File: 3-3.xls

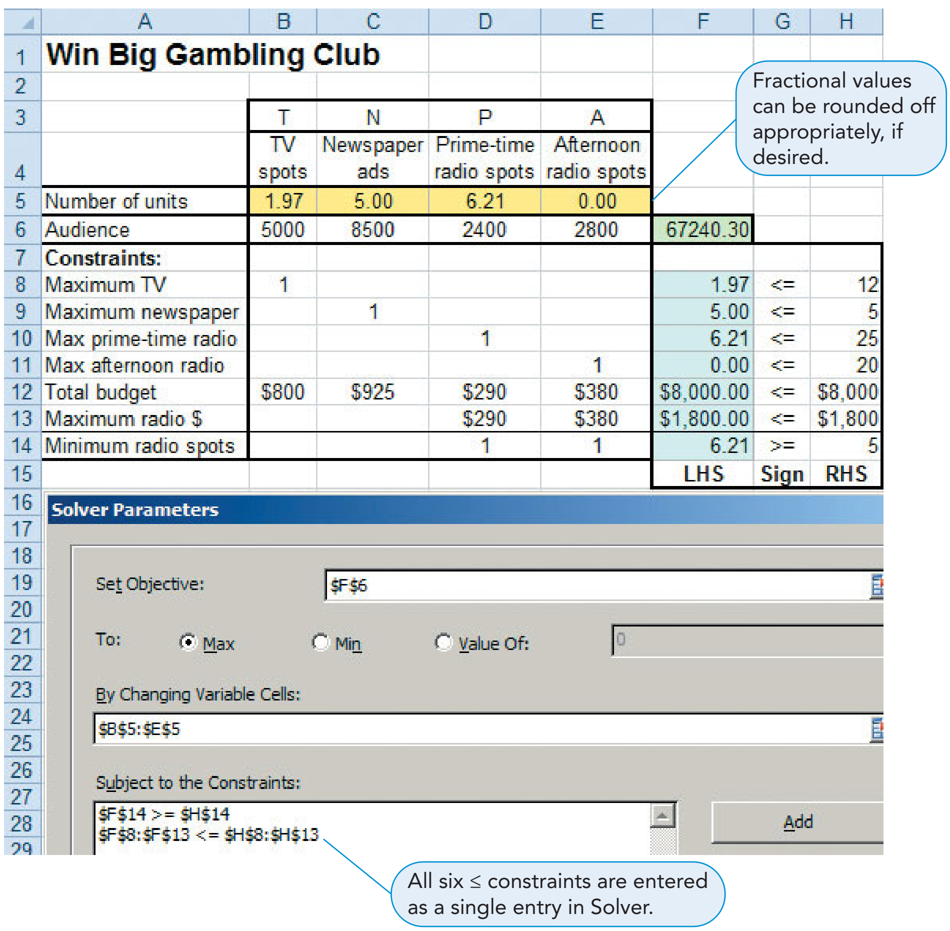
INTERPRETING THE RESULTS The optimal solution is found to be 1.97 television spots, 5 newspaper ads, 6.21 30-second prime time radio spots, and no 1-minute afternoon radio spots. This produces an audience exposure of 67,240.30 contacts. Here again, this solution turns out to have fractional values. Win Big would probably round down P to 6 spots and correspondingly round up T to 2 spots. A quick check indicates that the rounded solution satisfies all constraints even though T has been rounded up.

Marketing Research Problem

Selection of survey participants for consumer research is another popular use of LP.

LP has been applied to marketing research problems and the area of consumer research. The next example illustrates how LP can help statistical pollsters make strategic decisions.

SCREENSHOT 3-3
Excel Layout and Solver Entries for Win Big Gambling Club



	A	B	C	D	E	F	G	H
1	Win Big Gambling Club							
2								
3		T	N	P	A			
4		TV spots	Newspaper ads	Prime-time radio spots	Afternoon radio spots			
5	Number of units	1.97	5.00	6.21	0.00			
6	Audience	5000	8500	2400	2800	67240.30		
7	Constraints:							
8	Maximum TV	1				1.97	<=	12
9	Maximum newspaper		1			5.00	<=	5
10	Max prime-time radio			1		6.21	<=	25
11	Max afternoon radio				1	0.00	<=	20
12	Total budget	$800	$925	$290	$380	$8,000.00	<=	$8,000
13	Maximum radio $			$290	$380	$1,800.00	<=	$1,800
14	Minimum radio spots			1	1	6.21	>=	5
15						LHS	Sign	RHS

Management Sciences Associates (MSA) is a marketing and computer research firm based in Washington, DC, that handles consumer surveys. One of its clients is a national press service that periodically conducts political polls on issues of widespread interest. In a survey for the press service, MSA determines that it must fulfill several requirements in order to draw statistically valid conclusions on the sensitive issue of new U.S. immigration laws aimed at countering terrorism:

1. Survey at least 2,300 people in total in the United States.
2. Survey at least 1,000 people who are 30 years of age or younger.
3. Survey at least 600 people who are between 31 and 50 years of age.
4. Ensure that at least 15% of those surveyed live in a state that borders Mexico.
5. Ensure that at least 50% of those surveyed who are 30 years of age or younger live in a state that does not border Mexico.
6. Ensure that no more than 20% of those surveyed who are 51 years of age or over live in a state that borders Mexico.

MSA decides that all surveys should be conducted in person. It estimates that the costs of reaching people in each age and region category are as shown in Table 3.4. MSA's goal is to meet the six sampling requirements at the least possible cost.

FORMULATING THE PROBLEM The first step is to decide what the decision variables are. We note from Table 3.4 that people have been classified based on their ages (three categories) and regions (two categories). There is a separate cost associated with a person based on his or her age as well as his or her region. That is, for example, the cost for all persons from the ≤ 30 group is not the same and depends explicitly on whether the person is from a border state (\$7.50) or from a non-border state (\$6.90). The decision variables therefore need to identify each person surveyed based on his or her age as well as his or her region. Because we have three age categories and two regions, we need a total of 6 ($= 3 \times 2$) decision variables. We let

B_1 = number surveyed who are $\le$ 30 years of age and live in a border state
B_2 = number surveyed who are 31–50 years of age and live in a border state
B_3 = number surveyed who are $\ge$ 51 years of age and live in a border state
N_1 = number surveyed who are $\le$ 30 years of age and do not live in a border state
N_2 = number surveyed who are 31–50 years of age and do not live in a border state
N_3 = number surveyed who are $\ge$ 51 years of age and do not live in a border state

Objective function:

$$\text{Minimize total interview cost} = \$7.50B_1 + \$6.80B_2 + \$5.50B_3 + \$6.90N_1 + \$7.25N_2 + \$6.10N_3$$

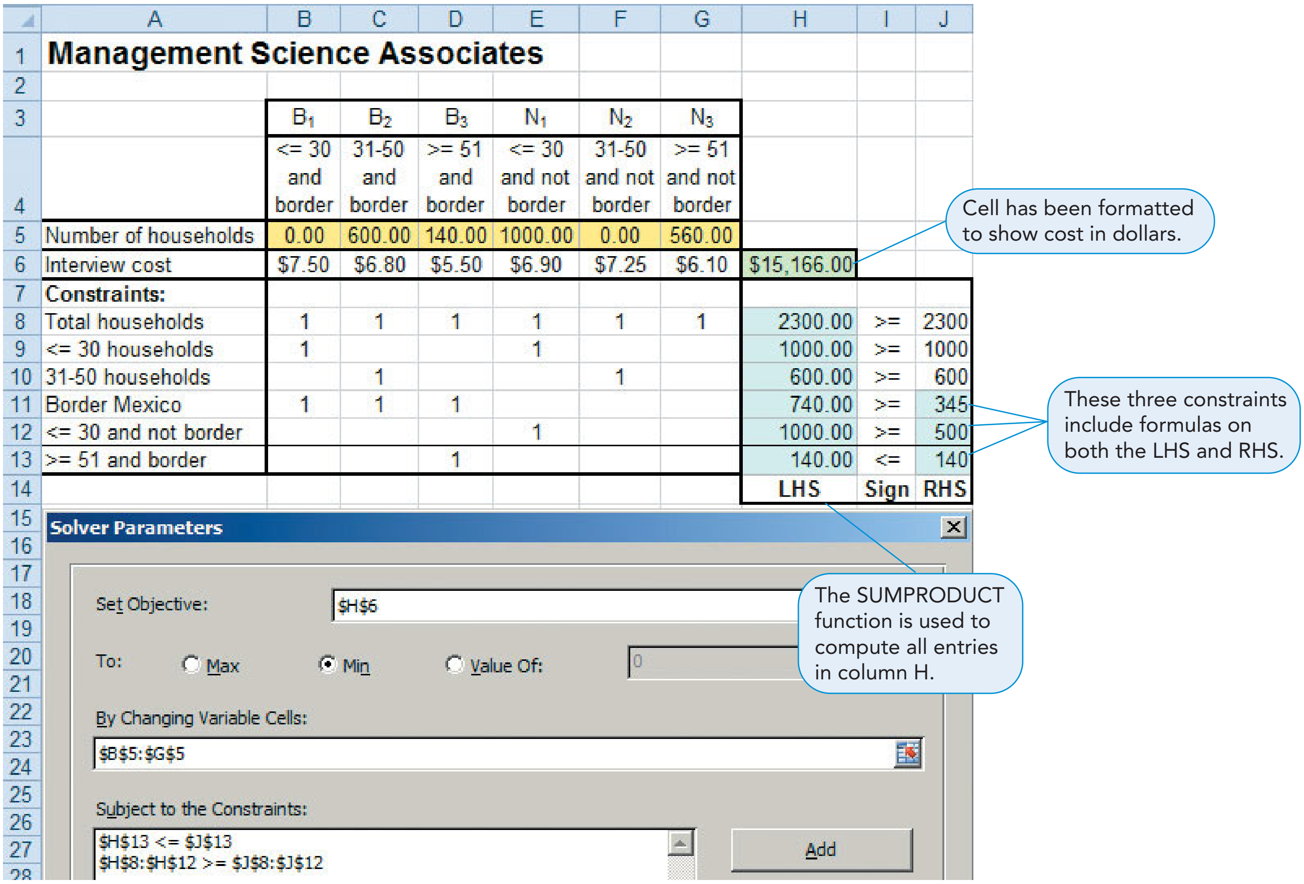
subject to the constraints

$$
\begin{array}{lll}
B_1 + B_2 + B_3 + N_1 + N_2 + N_3 \ge 2{,}300 & & \text{(total number surveyed)} \\
B_1 + N_1 & \ge 1{,}000 & \text{(persons 30 years or younger)} \\
B_2 + N_2 & \ge 600 & \text{(persons 31–50 in age)} \\
B_1 + B_2 + B_3 \ge 0.15(B_1 + B_2 + B_3 + N_1 + N_2 + N_3) & & \text{(border states)} \\
N_1 & \ge 0.5(B_1 + N_1) & (\le 30 \text{ years and not border state}) \\
B_3 & \le 0.2(B_3 + N_3) & \text{(51+ years and border state)} \\
B_1, B_2, B_3, N_1, N_2, N_3 & \ge 0 & \text{(nonnegativity)}
\end{array}
$$

TABLE 3.4
Data for Management Sciences Associates

	COST PER PERSON SURVEYED		
REGION	AGE $\le$ 30	AGE 31–50	AGE $\ge$ 51
State bordering Mexico	\$7.50	\$6.80	\$5.50
State not bordering Mexico	\$6.90	\$7.25	\$6.10

SCREENSHOT 3-4 Excel Layout and Solver Entries for Management Sciences Associates



SOLVING THE PROBLEM The Excel layout and Solver entries for this model are shown in Screenshot 3-4. In implementing this model for constraints 4 (border states), 5 ($\leq$30 years and not border state), and 6 (51+ years and border state), we have chosen to include Excel formulas for both the left-hand-side (LHS) and right-hand-side (RHS) entries. For example, for the border states constraint (4), cell H11 in Screenshot 3-4 represents the formula $(B_1 + B_2 + B_3)$ and is implemented in Excel by using the SUMPRODUCT function, as usual:

File: 3-4.xls

=SUMPRODUCT(B11:G11,$B$5:$G$5)

Cell J11 represents the formula $0.15(B_1 + B_2 + B_3 + N_1 + N_2 + N_3)$ in the RHS of constraint 4. For this cell, we can write the Excel formula as

=0.15*SUM(B5:G5) or =0.15*H8

To be consistent with our layout format, we have colored both cells H11 and J11 blue to indicate the presence of Excel formulas in both cells. In a similar fashion, the formula in cells J12 (RHS for constraint 5) and J13 (RHS for constraint 6) would be

Cell J12: =0.5*(B5+E5)
Cell J13: =0.2*(D5+G5)

If desired, constraints can be algebraically modified to bring all variables to the LHS.

Alternatively, if we prefer, we can algebraically modify constraints 4, 5, and 6 to bring all the variables to the LHS and just leave a constant on the RHS. For example, constraint 4 is presently modeled as

$$B_1 + B_2 + B_3 \geq 0.15(B_1 + B_2 + B_3 + N_1 + N_2 + N_3)$$

This can be rewritten as

$$B_1 + B_2 + B_3 - 0.15(B_1 + B_2 + B_3 + N_1 + N_2 + N_3) \geq 0$$

IN ACTION **Nestlé Uses Linear Programming to Improve Financial Reporting**

Nestlé's executive information system (EIS) department provides top management with operational, financial, and strategic information gathered from the firm's subsidiaries. In an effort to improve its service, the EIS department developed four business-analytics modules that use decision modeling techniques including linear programming. These four modules are integrated within a financial analysis framework that financial managers can use as needed to evaluate the economic profitability of new projects and develop strategies for its subsidiaries.

This approach has been effectively disseminated within the Nestlé group resulting in an increasing awareness among managers of the usefulness of decision modeling techniques. Interestingly, the linear programming module at Nestle focuses on the use of Solver, its answer report, and its sensitivity report–topics that are discussed in detail in Chapters 2–4 of this textbook.

Source: Based on C. Oggier, E. Fragnière, and J. Stuby. "Nestlé Improves Its Financial Reporting with Management Science," *Interfaces* 35, 4 (July–August 2005): 271–280.

which simplifies to

$$0.85B_1 + 0.85B_2 + 0.85B_3 - 0.15N_1 - 0.15N_2 - 0.15N_3 \geq 0$$

Likewise, constraints 5 and 6 would be

Constraint 5: $-0.5B_1 + 0.5N_1 \geq 0$ ($\leq$30 years and not border state)

Constraint 6: $0.8B_3 - 0.2N_3 \leq 0$ (51+ years and border state)

Algebraic modifications of constraints are not required for implementation in Excel.

Note that such algebraic manipulations are not required to implement this model in Excel. In fact, in many cases it is probably not preferable to make such modifications because the meaning of the revised constraint is not intuitively obvious. For example, the coefficient 0.85 for B_1 in the modified form for constraint 4 has no direct significance or meaning in the context of the problem.

INTERPRETING THE RESULTS The optimal solution to MSA's marketing research problem costs $15,166 and requires the firm to survey people as follows:

People who are 31–50 years of age and live in a border state = 600

People who are $\geq$ 51 years of age and live in a border state = 140

People who are $\leq$ 30 years of age and do not live in a border state = 1,000

People who are $\geq$ 51 years of age and do not live in a border state = 560

3.4 Finance Applications

Portfolio Selection Problem

A problem frequently encountered by managers of banks, mutual funds, investment services, and insurance companies is the selection of specific investments from among a wide variety of alternatives. A manager's overall objective is usually to maximize expected return on investment, given a set of legal, policy, or risk constraints.

Maximizing return on investment subject to a set of risk constraints is a popular financial application of LP.

Consider the example of International City Trust (ICT), which invests in trade credits, corporate bonds, precious metal stocks, mortgage-backed securities, and construction loans. ICT has $5 million available for immediate investment and wishes to maximize the interest earned on its investments over the next year. The specifics of the investment possibilities are shown in Table 3.5. For each type of investment, the table shows the expected return over the next year as well as a score that indicates the risk associated with the investment. (A lower score implies less risk.)

To encourage a diversified portfolio, the board of directors has placed several limits on the amount that can be committed to any one type of investment: (1) No more than 25% of the total amount invested may be in any single type of investment, (2) at least 30% of the funds invested must be in precious metals, (3) at least 45% must be invested in trade credits and corporate bonds, and (4) the average risk score of the total investment must be 2 or less.

TABLE 3.5
Data for International City Trust

INVESTMENT	INTEREST EARNED	RISK SCORE
Trade credits	7%	1.7
Corporate bonds	10%	1.2
Gold stocks	19%	3.7
Platinum stocks	12%	2.4
Mortgage securities	8%	2.0
Construction loans	14%	2.9

Decision variables in financial planning models usually define the amounts to be invested in each investment choice.

FORMULATING THE PROBLEM The decision variables in most investment planning problems correspond to the amount that should be invested in each investment choice. In ICT's case, to model the investment decision as an LP problem, we let

$$
\begin{aligned}
T &= \text{dollars invested in trade credit}\\
B &= \text{dollars invested in corporate bonds}\\
G &= \text{dollars invested in gold stocks}\\
P &= \text{dollars invested in platinum stocks}\\
M &= \text{dollars invested in mortgage securities}\\
C &= \text{dollars invested in construction loans}
\end{aligned}
$$

The objective function may then be written as

$$\text{Maximize dollars of interest earned} = 0.07T + 0.10B + 0.19G + 0.12P + 0.08M + 0.14C$$

The constraints control the amounts that may be invested in each type of investment. ICT has \$5 million available for investment. Hence the first constraint is

$$T + B + G + P + M + C \leq 5{,}000{,}000$$

Could we express this constraint by using the $=$ sign rather than $\leq$? Because any dollar amount that is not invested by ICT does not earn any interest, it is logical to expect that, under normal circumstances, ICT would invest the entire \$5 million. However, it is possible in some unusual cases that the conditions placed on investments by the board of directors are so restrictive that it is not possible to find an investment strategy that allows the entire amount to be invested. To guard against this possibility, it is preferable to write the preceding constraint using the $\leq$ sign. Note that if the investments conditions do permit the entire amount to be invested, the optimal solution will automatically do so because this is a maximization problem.

The other constraints are

$$
\begin{aligned}
T &\leq 0.25\,(T + B + G + P + M + C)\\
B &\leq 0.25\,(T + B + G + P + M + C)\\
G &\leq 0.25\,(T + B + G + P + M + C)\\
P &\leq 0.25\,(T + B + G + P + M + C)\\
M &\leq 0.25\,(T + B + G + P + M + C)\\
C &\leq 0.25\,(T + B + G + P + M + C)
\end{aligned}
$$

If we are sure that the investment conditions will permit the entire \$5 million to be invested, we can write the RHS of each of these constraints as a constant \$1,250,000 ($=0.25 \times \$5{,}000{,}000$). However, for the same reason discussed for the first constraint, it is preferable to write the RHS for these constraints (as well as all other constraints in this problem) in terms of the sum of the decision variables rather than as constants.

As in the marketing research problem, we can, if we wish, algebraically modify these constraints so that all variables are on the LHS and only a constant is on the RHS. (*Note:* Such algebraic manipulations are not required for Excel to solve these problems, and we do not recommend that they be done.) For example, the constraint regarding trade credits would then be written as

$$0.75T - 0.25B - 0.25G - 0.25P - 0.25M - 0.25C \leq 0 \qquad \text{(no more than 25\% in trade credits)}$$

The constraints that at least 30% of the funds invested must be in precious metals and that at least 45% must be invested in trade credits and corporate bonds are written, respectively, as

$$G + P \geq 0.30\,(T + B + G + P + M + C)$$
$$T + B \geq 0.45\,(T + B + G + P + M + C)$$

Next, we come to the risk constraint which states that the average risk score of the total investment must be 2 or less. To calculate the average risk score, we need to take the weighted sum of the risk and divide it by the total amount invested. This constraint may be written as

$$\frac{1.7T + 1.2B + 3.7G + 2.4P + 2.0M + 2.9C}{T + B + G + P + M + C} \leq 2$$

Excel Notes

- Although the previous expression for the risk constraint is a valid linear equation, Excel's Solver sometimes misinterprets the division sign in the expression as an indication of nonlinearity and gives the erroneous message "The linearity conditions required by this LP Solver are not satisfied."
- Likewise, if Solver sets all decision variable values to zero at any stage during the solution process, the denominator in the previous expression becomes zero. This causes a division-by-zero error, which could sometimes lead to a "Solver encountered an error value in the Objective Cell or a Constraint cell" message.
- If a constraint with a denominator that is a function of the decision variables causes either of these situations to occur in your model, we recommend that you algebraically modify the constraint before implementing it in Excel. For example, in ICT's problem, the risk constraint could be implemented as

$$1.7T + 1.2B + 3.7G + 2.4P + 2.0M + 2.9C \leq 2(T + B + G + P + M + C)$$

Finally, we have the nonnegativity constraints:

$$T, B, G, P, M, C \geq 0$$

File: 3-5.xls, sheet: 3-5A

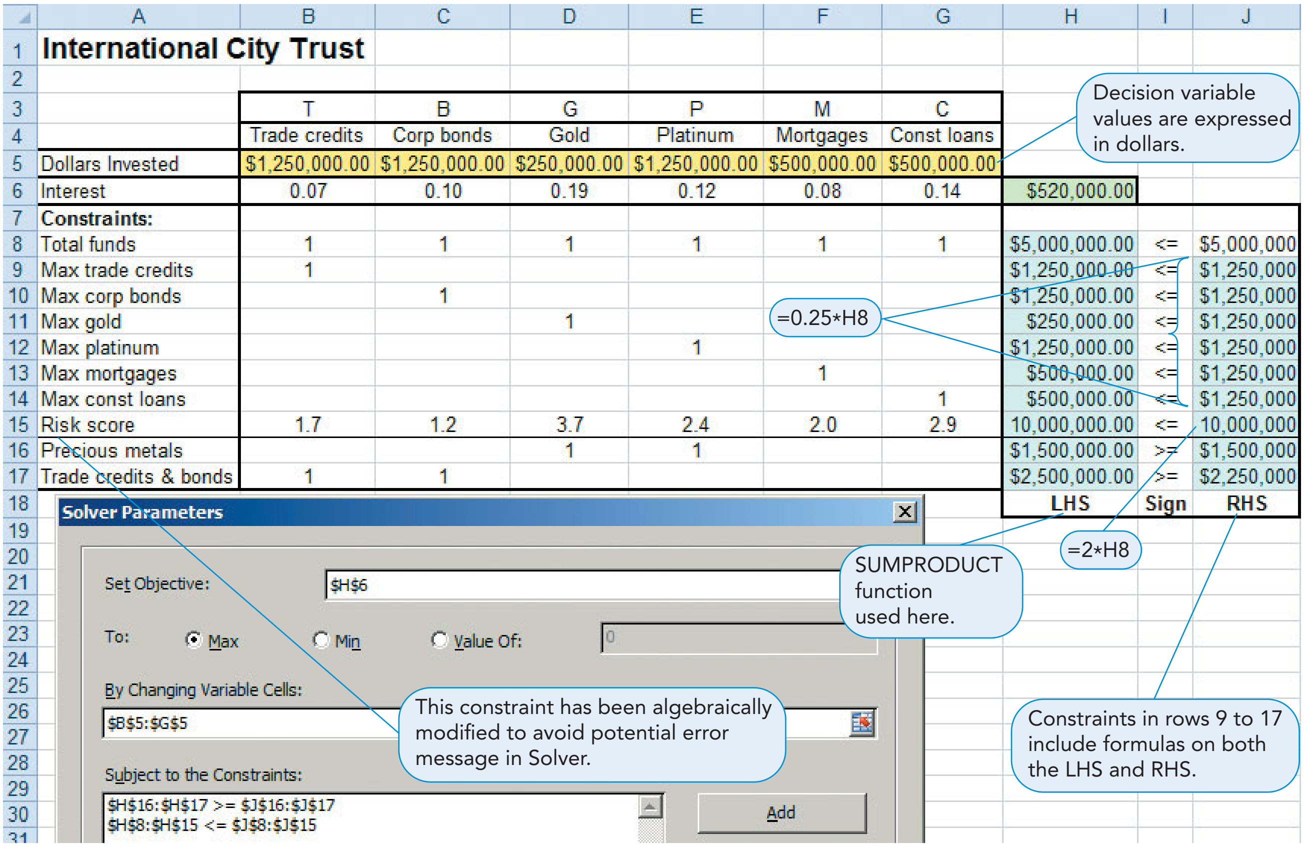
SOLVING THE PROBLEM The Excel layout and Solver entries for this model are shown in Screenshot 3-5A. Because we have chosen to express the RHS values for most constraints in terms of the decision variables rather than as constants, we have formulas on both sides for all but one of the constraints. The only exception is the constraint which specifies that \$5 million is available for investment. As usual, cells representing the LHS values of constraints (cells H8:H17) use the SUMPRODUCT function. Cells representing RHS values are modeled using simple Excel formulas, as in the marketing research problem. For example, the RHS (cell J9) for the constraint specifying no more than 25% in trade credits would contain the formula

=0.25*SUM(B5:G5) or =0.25*H8

Likewise, the RHS (cell J15) for the risk score constraint would contain the formula =2*H8 to reflect the algebraic manipulation of this constraint (see *Excel Notes* above).

INTERPRETING THE RESULTS The optimal solution is to invest \$1.25 million each in trade credits, corporate bonds, and platinum stocks; \$500,000 each in mortgage-backed securities and construction loans; and \$250,000 in gold stocks—earning a total interest of \$520,000. In this case, the investment conditions do allow the entire \$5 million to be invested, meaning that we could have expressed the RHS values of all constraints by using constants without affecting the optimal solution.

SCREENSHOT 3-5A **Excel Layout and Solver Entries for International City Trust**



Alternate Formulations of the Portfolio Selection Problem

Alternate formulations are possible for some LP problems.

In our discussion of the portfolio selection problem, we have chosen to express the decision variables as the number of dollars invested in each choice. However, for this problem (as well as many other problems), we can define the decision variables in alternate fashions. We address two alternatives here.

Problems with large variability in the magnitudes of parameter and/or variable values should be scaled.

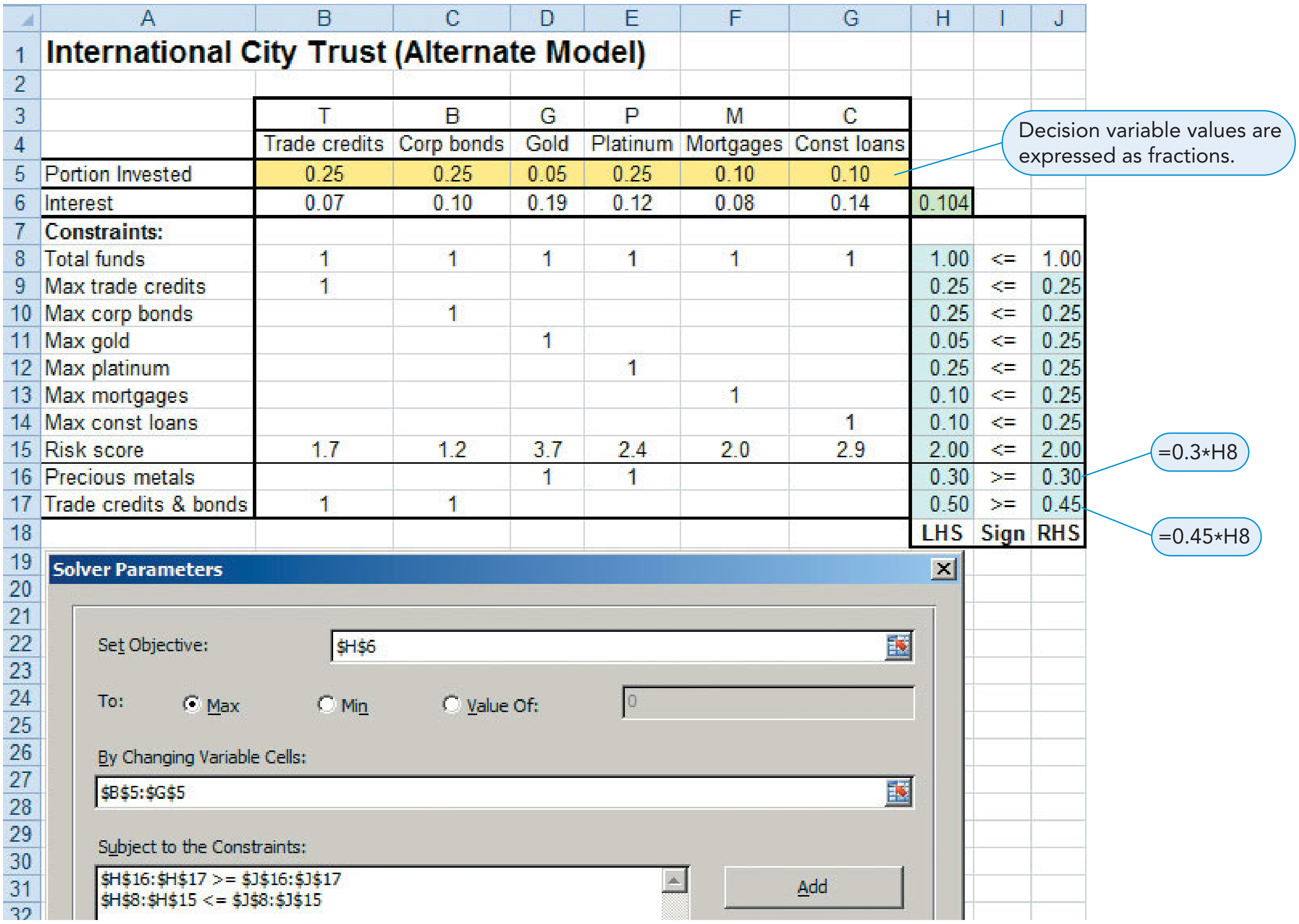
First, we could set up the decision variables to represent the number of dollars invested in *millions*. As noted in Chapter 2, it is usually a good idea in practice to scale problems in which values of the objective function coefficients and constraint coefficients of different constraints differ by several orders of magnitude. One way to do so would be to click the Use Automatic Scaling option available in Solver, discussed in Chapter 2. In ICT's problem, observe that the only impact of this revised definition of the decision variables on the written formulation would be to replace the $5,000,000 in the total funds available constraint by $5. Likewise, cell J8 in Screenshot 3-5A would show a value of $5. Because all other RHS values are functions of the decision variables, they will automatically reflect the revised situation. The optimal solution will show values of $1.25 each for trade credits, corporate bonds, and platinum stocks; $0.50 each for mortgage-backed securities and construction loan; and $0.25 for gold stocks—earning a total interest of $0.52.

Second, if we assume that the $5 million represents 100% of the money available, we could define the decision variables as the portion (or percentage) of this amount invested in each investment choice. For example, we could define

$$T = \text{portion (or percentage) of the \$5 million invested in trade credit}$$

We could do likewise for the other five decision variables (B, G, P, M, and C). The interesting point to note in this case is that the actual amount available ($5 million) is not really relevant and does not figure anywhere in the formulation. Rather, the idea is that if we can decide how to distribute 100% of the funds available among the various choices, we can apply those percentage allocations to any available amount.

SCREENSHOT 3-5B **Excel Layout and Solver Entries for International City Trust—Alternate Model**



	A	B	C	D	E	F	G	H	I	J
1	International City Trust (Alternate Model)									
2										
3		T	B	G	P	M	C			
4		Trade credits	Corp bonds	Gold	Platinum	Mortgages	Const loans			
5	Portion Invested	0.25	0.25	0.05	0.25	0.10	0.10			
6	Interest	0.07	0.10	0.19	0.12	0.08	0.14	0.104		
7	Constraints:									
8	Total funds	1	1	1	1	1	1	1.00	<=	1.00
9	Max trade credits	1						0.25	<=	0.25
10	Max corp bonds		1					0.25	<=	0.25
11	Max gold			1				0.05	<=	0.25
12	Max platinum				1			0.25	<=	0.25
13	Max mortgages					1		0.10	<=	0.25
14	Max const loans						1	0.10	<=	0.25
15	Risk score	1.7	1.2	3.7	2.4	2.0	2.9	2.00	<=	2.00
16	Precious metals			1	1			0.30	>=	0.30
17	Trade credits & bonds	1	1					0.50	>=	0.45
18								LHS	Sign	RHS

File: 3-5.xls, sheet: 3-5B

The revised Excel layout and solution for this problem is shown in Screenshot 3-5B. In ICT's problem, we note that the only impact of this revised definition of decision variables is to replace the $5 million in the total funds available constraint by 1 (or 100%). Other constraint RHS values (cells J9:J14 and J16:J17) no longer show dollar values but instead show total portions invested. The RHS for the constraint in row 15 (cell J15) shows the average risk score of the investment strategy. Screenshot 3-5B indicates that the optimal solution now shows 0.25 each for trade credits, corporate bonds, and platinum stocks; 0.10 each for mortgage-backed securities and construction loans; and 0.05 for gold stocks—earning a total return of 0.104. If we multiply each of these portions by the amount available ($5 million), we get the same dollar values as in the solution to the original formulation (Screenshot 3-5A).

It is important to define the decision variables in an LP problem as precisely as possible.

The preceding discussion reinforces the need to define the decision variables in an LP problem as precisely as possible. As we observed, depending on how the decision variables have been defined, the constraints and, hence, the resulting optimal solution values will be different.

3.5 Employee Staffing Applications

Labor Planning Problem

Labor staffing is a popular application of LP, especially when there is wide fluctuation in labor needs between various periods.

Labor planning problems address staffing needs over a specific planning horizon, such as a day, week, month, or year. They are especially useful when staffing needs are different during different time periods in the planning horizon and managers have some flexibility in assigning workers to jobs that require overlapping or interchangeable talents. Large banks frequently use LP to tackle their labor staffing problem.

Hong Kong Bank of Commerce and Industry is a busy bank that has requirements for between 10 and 18 tellers, depending on the time of day. The afternoon time, from noon to

TABLE 3.6
Tellers Required for Hong Kong Bank

TIME PERIOD	NUMBER REQUIRED
9 A.M.–10 A.M.	10
10 A.M.–11 A.M.	12
11 A.M.–Noon	14
Noon–1 P.M.	16
1 P.M.–2 P.M.	18
2 P.M.–3 P.M.	17
3 P.M.–4 P.M.	15
4 P.M.–5 P.M.	10

3 P.M., is usually heaviest. Table 3.6 indicates the workers needed at various hours that the bank is open.

The bank now employs 12 full-time tellers but also has several people available on its roster of part-time employees. A part-time employee must put in exactly 4 hours per day but can start anytime between 9 A.M. and 1 P.M. Part-timers are a fairly inexpensive labor pool because no retirement or lunch benefits are provided for them. Full-timers, on the other hand, work from 9 A.M. to 5 P.M. but are allowed 1 hour for lunch. (Half of the full-timers eat at 11 A.M. and the other half at noon.) Each full-timer thus provides 35 hours per week of productive labor time.

The bank's corporate policy limits part-time hours to a maximum of 50% of the day's total requirement. Part-timers earn \$7 per hour (or \$28 per day) on average, and full-timers earn \$90 per day in salary and benefits, on average. The bank would like to set a schedule that would minimize its total personnel costs. It is willing to release one or more of its full-time tellers if it is cost-effective to do so.

FORMULATING AND SOLVING THE PROBLEM In employee staffing problems, we typically need to determine how many employees need to start their work at the different starting times permitted. For example, in Hong Kong Bank's case, we have full-time tellers who all start work at 9 A.M. and part-timers who can start anytime between 9 A.M. and 1 P.M. Let

F = number of full-time tellers to use (all starting at 9 A.M.)
P_1 = number of part-timers to use, starting at 9 A.M. (leaving at 1 P.M.)
P_2 = number of part-timers to use, starting at 10 A.M. (leaving at 2 P.M.)
P_3 = number of part-timers to use, starting at 11 A.M. (leaving at 3 P.M.)
P_4 = number of part-timers to use, starting at noon (leaving at 4 P.M.)
P_5 = number of part-timers to use, starting at 1 P.M. (leaving at 5 P.M.)

The objective function is

$$\text{Minimize total daily personnel cost} = \$90F + \$28(P_1 + P_2 + P_3 + P_4 + P_5)$$

Next, we write the constraints. For each hour, the available number of tellers must be at least equal to the required number of tellers. This is a simple matter of counting which of the different employees (defined by the decision variables) are working during a given time period and which are not. It is also important to remember that half the full-time tellers break for lunch between 11 A.M. and noon and the other half break between noon and 1 P.M.:

$$
\begin{array}{rll}
F + P_1 & \geq 10 & \text{(9 A.M.–10 A.M. requirement)} \\
F + P_1 + P_2 & \geq 12 & \text{(10 A.M.–11 A.M. requirement)} \\
0.5F + P_1 + P_2 + P_3 & \geq 14 & \text{(11 A.M.–12 noon requirement)} \\
0.5F + P_1 + P_2 + P_3 + P_4 & \geq 16 & \text{(12 noon–1 P.M. requirement)} \\
F + P_2 + P_3 + P_4 + P_5 & \geq 18 & \text{(1 P.M.–2 P.M. requirement)} \\
F + P_3 + P_4 + P_5 & \geq 17 & \text{(2 P.M.–3 P.M. requirement)} \\
F + P_4 + P_5 & \geq 15 & \text{(3 P.M.–4 P.M. requirement)} \\
F + P_5 & \geq 10 & \text{(4 P.M.–5 P.M. requirement)}
\end{array}
$$

Only 12 full-time tellers are available, so

$$F \le 12$$

Part-time worker hours cannot exceed 50% of total hours required each day, which is the sum of the tellers needed each hour. Hence

$$4(P_1 + P_2 + P_3 + P_4 + P_5) \le 0.5(10 + 12 + 14 + 16 + 18 + 17 + 15 + 10)$$

or

$$\begin{aligned} 4P_1 + 4P_2 + 4P_3 + 4P_4 + 4P_5 &\le 56 \\ F, P_1, P_2, P_3, P_4, P_5 &\ge 0 \end{aligned}$$

File: 3-6.xls

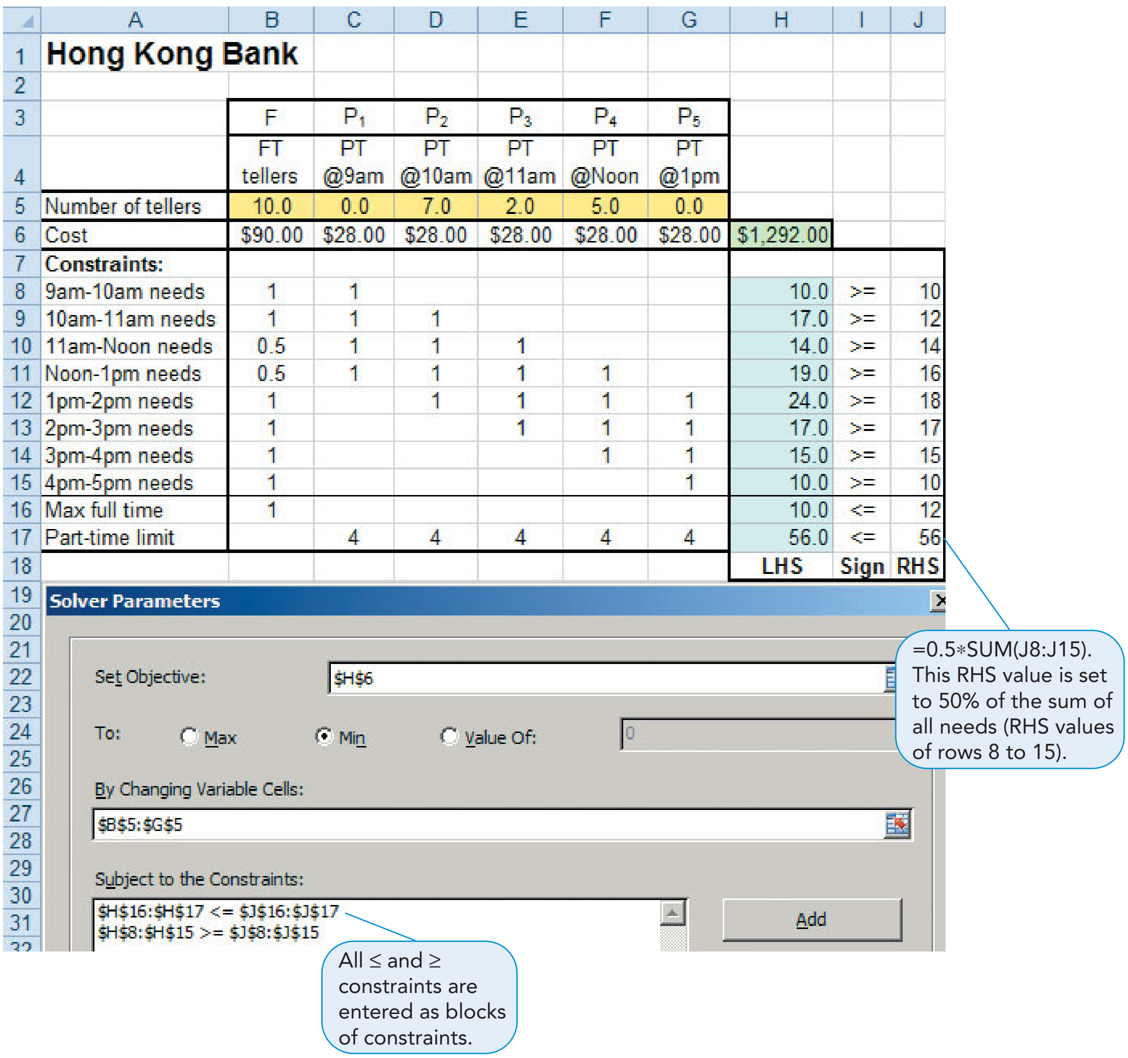
The Excel layout and Solver entries for this model are shown in Screenshot 3-6.

INTERPRETING THE RESULTS Screenshot 3-6 reveals that the optimal solution is to employ 10 full-time tellers, 7 part-time tellers at 10 A.M., 2 part-time tellers at 11 A.M., and 5 part-time tellers at noon, for a total cost of $1,292 per day. Because we are using only 10 of the 12 available full-time tellers, Hong Kong Bank can choose to release up to 2 of the full-time tellers.

Alternate optimal solutions are common in many LP applications.

It turns out that there are several alternate optimal solutions that Hong Kong Bank can employ. In practice, the sequence in which you present constraints in a model can affect the specific solution that is found. We revisit this example in Chapter 4 (Solved Problem 4-1) to

SCREENSHOT 3-6 Excel Layout and Solver Entries for Hong Kong Bank



	A	B	C	D	E	F	G	H	I	J
1	Hong Kong Bank									
2										
3		F	P_1	P_2	P_3	P_4	P_5			
4		FT tellers	PT @9am	PT @10am	PT @11am	PT @Noon	PT @1pm			
5	Number of tellers	10.0	0.0	7.0	2.0	5.0	0.0			
6	Cost	$90.00	$28.00	$28.00	$28.00	$28.00	$28.00	$1,292.00		
7	Constraints:									
8	9am-10am needs	1	1					10.0	>=	10
9	10am-11am needs	1	1	1				17.0	>=	12
10	11am-Noon needs	0.5	1	1	1			14.0	>=	14
11	Noon-1pm needs	0.5	1	1	1	1		19.0	>=	16
12	1pm-2pm needs	1		1	1	1	1	24.0	>=	18
13	2pm-3pm needs	1			1	1	1	17.0	>=	17
14	3pm-4pm needs	1				1	1	15.0	>=	15
15	4pm-5pm needs	1					1	10.0	>=	10
16	Max full time	1						10.0	<=	12
17	Part-time limit		4	4	4	4	4	56.0	<=	56
18								LHS	Sign	RHS

study how we can use the Sensitivity Report generated by Solver to detect and identify alternate optimal solutions.

For this problem, one alternate solution is to employ 10 full-time tellers, 6 part-time tellers at 9 A.M., 1 part-time teller at 10 A.M., 2 part-time tellers at 11 A.M., and 5 part-time tellers at noon. The cost of this policy is also $1,292.

Note that we are setting up the teller requirement constraints as ≥ constraints rather than as = constraints. The reason for this should be obvious by now: If we try to *exactly* satisfy the teller requirements every period, the fact that each teller (full or part time) works more than one hour at a stretch may make it impossible to simultaneously satisfy all requirements as = constraints. In fact, if you replace the ≥ sign with = for the teller requirement constraints in Hong Kong Bank's model (rows 8 to 15) and resolve the problem, Solver returns a "Solver could not find a feasible solution" message.

Extensions to the Labor Planning Problem

The previous example considered the labor requirements during different time periods of a single day. In other labor planning problems, the planning horizon may consist of a week, a month, or a year. In this case, the decision variables will correspond to the different work schedules that workers can follow and will denote the number of workers who should follow a specific work schedule. Solved Problem 3-1 at the end of this chapter illustrates an example where the planning horizon is a week and the time periods are days of the week. The worker requirements are specified for each of the seven days of the week.

Labor staffing problems involving multiple shifts per period can be modeled by using LP.

In extended versions of this problem, the time periods may also include specific shifts. For example, if there are two shifts per day (day shift and night shift), there are then 14 time periods (seven days of the week, two shifts per day) in the problem. Worker requirements need to be specified for each of these 14 time periods. In this case, the work schedules need to specify the exact days and shifts that the schedule denotes. For example, an available work schedule could be "Monday to Friday on shift 1, Saturday and Sunday off." Another available work schedule could be "Tuesday to Saturday on shift 2, Sunday and Monday off." The decision analyst would need to specify all available work schedules before setting up the problem as an LP model.

Assignment Problem

Assigning people to jobs, jobs to machines, and so on is an application of LP called the assignment problem.

Assignment problems involve determining the most efficient assignment of people to jobs, machines to tasks, police cars to city sectors, salespeople to territories, and so on. The assignments are done on a one-to-one basis. For example, in a people-to-jobs assignment problem, each person is assigned to exactly one job, and, conversely, each job is assigned to exactly one person. Fractional assignments are not permitted. The objective might be to minimize the total cost of the assignments or maximize the total effectiveness or benefit of the assignments.

The assignment problem is an example of a special type of LP problem known as a network flow problem, and we study this type of problem in greater detail in Chapter 5.

3.6 Transportation Applications

Vehicle Loading Problem

Vehicle loading problems involve deciding which items to load on a vehicle (e.g. truck, ship, aircraft) to maximize the total value of the load shipped. The items loaded may need to satisfy several constraints, such as weight and volume limits of the vehicle, minimum levels of certain items that may need to be accepted, etc. As an example, we consider Goodman Shipping, an Orlando firm owned by Steven Goodman. One of his trucks, with a weight capacity of 15,000 pounds and a volume capacity of 1,300 cubic feet, is about to be loaded. Awaiting shipment are the items shown in Table 3.7. Each of these six items, we see, has an associated total dollar value, available weight, and volume per pound that the item occupies. The objective is to maximize the total value of the items loaded onto the truck without exceeding the truck's weight and volume capacities.

FORMULATING AND SOLVING THE PROBLEM The decision variables in this problem define the number of pounds of each item that should be loaded on the truck. There would be six decision

TABLE 3.7
Shipments for Goodman Shipping

ITEM	VALUE	WEIGHT (POUNDS)	VOLUME (CU. FT. PER POUND)
1	$15,500	5,000	0.125
2	$14,400	4,500	0.064
3	$10,350	3,000	0.144
4	$14,525	3,500	0.448
5	$13,000	4,000	0.048
6	$ 9,625	3,500	0.018

variables (one for each item) in the model. In this case, the dollar value of each item needs to be appropriately scaled for use in the objective function. For example, if the total value of the 5,000 pounds of item 1 is $15,500, the value per pound is then 3.10 (=15,500/5,000 pounds). Similar calculations can be made for the other items to be shipped.

Let W_i be the weight (in pounds) of each item i loaded on the truck. The LP model can then be formulated as follows:

$$\text{Maximize load value} = \$3.10W_1 + \$3.20W_2 + \$3.45W_3 + \$4.15W_4 + \$3.25W_5 + \$2.75W_6$$

subject to the constraints

$$\begin{array}{lrl}
W_1 + W_2 + W_3 + W_4 + W_5 + W_6 & \leq 15{,}000 & \text{(weight limit of truck)} \\
0.125\,W_1 + 0.064\,W_2 + 0.144\,W_3 + & & \\
\quad 0.0448\,W_4 + 0.048\,W_5 + 0.018\,W_6 & \leq 1{,}300 & \text{(volume limit of truck)} \\
W_1 & \leq 5{,}000 & \text{(item 1 availability)} \\
W_2 & \leq 4{,}500 & \text{(item 2 availability)} \\
W_3 & \leq 3{,}000 & \text{(item 3 availability)} \\
W_4 & \leq 3{,}500 & \text{(item 4 availability)} \\
W_5 & \leq 4{,}000 & \text{(item 5 availability)} \\
W_6 & \leq 3{,}500 & \text{(item 6 availability)} \\
W_1, W_2, W_3, W_4, W_5, W_6 & \geq 0 & \text{(nonnegativity)}
\end{array}$$

Screenshot 3-7A shows the Excel layout and Solver entries for Goodman Shipping's LP model.

File: 3-7.xls, sheet: 3-7A

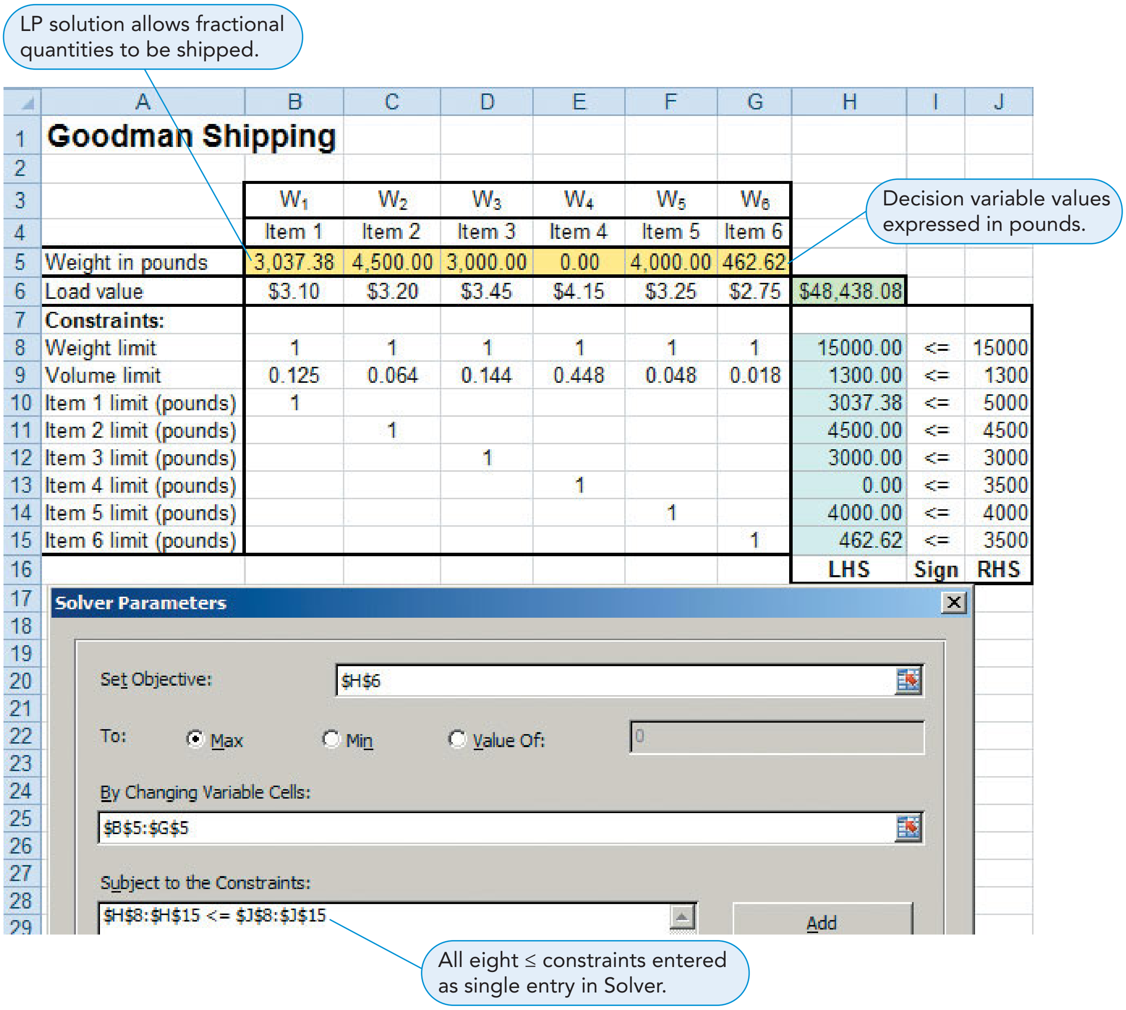
INTERPRETING THE RESULTS The optimal solution in Screenshot 3-7A yields a total value of $48,438.08 and requires Goodman to ship 3,037.38 pounds of item 1; 4,500 pounds of item 2; 3,000 pounds of item 3; 4,000 pounds of item 5; and 462.62 pounds of item 6. The truck is fully loaded from both weight and volume perspectives. As usual, if shipments have to be integers, Goodman can probably round down the totals for items 1 and 6 without affecting the total dollar value too much. Interestingly, the only item that is not included for loading is item 4, which has the highest dollar value per pound. However, its relatively high volume (per pound loaded) makes it an unattractive item to load.

File: 3-7.xls, sheet: 3-7B

ALTERNATE FORMULATIONS As in the portfolio selection problem we studied previously, there are alternate ways in which we could define the decision variables for this problem. For example, the decision variables could denote the portion (or percentage) of each item that is accepted for loading. Under this approach, let P_i be the portion of each item i loaded on the truck. Screenshot 3-7B on page 87 shows the Excel layout and solution for the alternate model, using these revised decision variables. The layout for this model is identical to that shown in Screenshot 3-7A, and you should be able to recognize its written formulation easily. The coefficients in the volume constraint show the volume occupied by the entire quantity of an item. For example, the volume coefficient for item 1 is 625 cubic feet (= 0.125 cubic feet per pound × 5,000 pounds). Likewise, the coefficient for item 2 is 288 cubic feet (= 0.064 cubic feet per pound × 4,500 pounds).

The final six constraints reflect the fact that at most one "unit" (i.e., a proportion of 1) of an item can be loaded onto the truck. In effect, if Goodman can load a *portion* of an item (e.g., item

SCREENSHOT 3-7A Excel Layout and Solver Entries for Goodman Shipping



	A	B	C	D	E	F	G	H	I	J
1	Goodman Shipping									
2										
3		W_1	W_2	W_3	W_4	W_5	W_6			
4		Item 1	Item 2	Item 3	Item 4	Item 5	Item 6			
5	Weight in pounds	3,037.38	4,500.00	3,000.00	0.00	4,000.00	462.62			
6	Load value	$3.10	$3.20	$3.45	$4.15	$3.25	$2.75	$48,438.08		
7	Constraints:									
8	Weight limit	1	1	1	1	1	1	15000.00	<=	15000
9	Volume limit	0.125	0.064	0.144	0.448	0.048	0.018	1300.00	<=	1300
10	Item 1 limit (pounds)	1						3037.38	<=	5000
11	Item 2 limit (pounds)		1					4500.00	<=	4500
12	Item 3 limit (pounds)			1				3000.00	<=	3000
13	Item 4 limit (pounds)				1			0.00	<=	3500
14	Item 5 limit (pounds)					1		4000.00	<=	4000
15	Item 6 limit (pounds)						1	462.62	<=	3500
16								LHS	Sign	RHS

1 is a batch of 1,000 folding chairs, not all of which need to be shipped together), the proportions P_i will all have values ranging from 0 (none of that item is loaded) to 1 (all of that item is loaded).

The solution to this model shows that the maximum load value is $48,438.08. This load value is achieved by shipping 60.748% of item 1 ($0.60748 \times 5{,}000 = 3{,}037.4$ pounds) of item 1; all available quantities (i.e., 100%) of items 2, 3, and 5; and 13.218% (462.63 pounds) of item 6. As expected, this is the same solution we obtained in Screenshot 3-7A, using the original model for this problem.

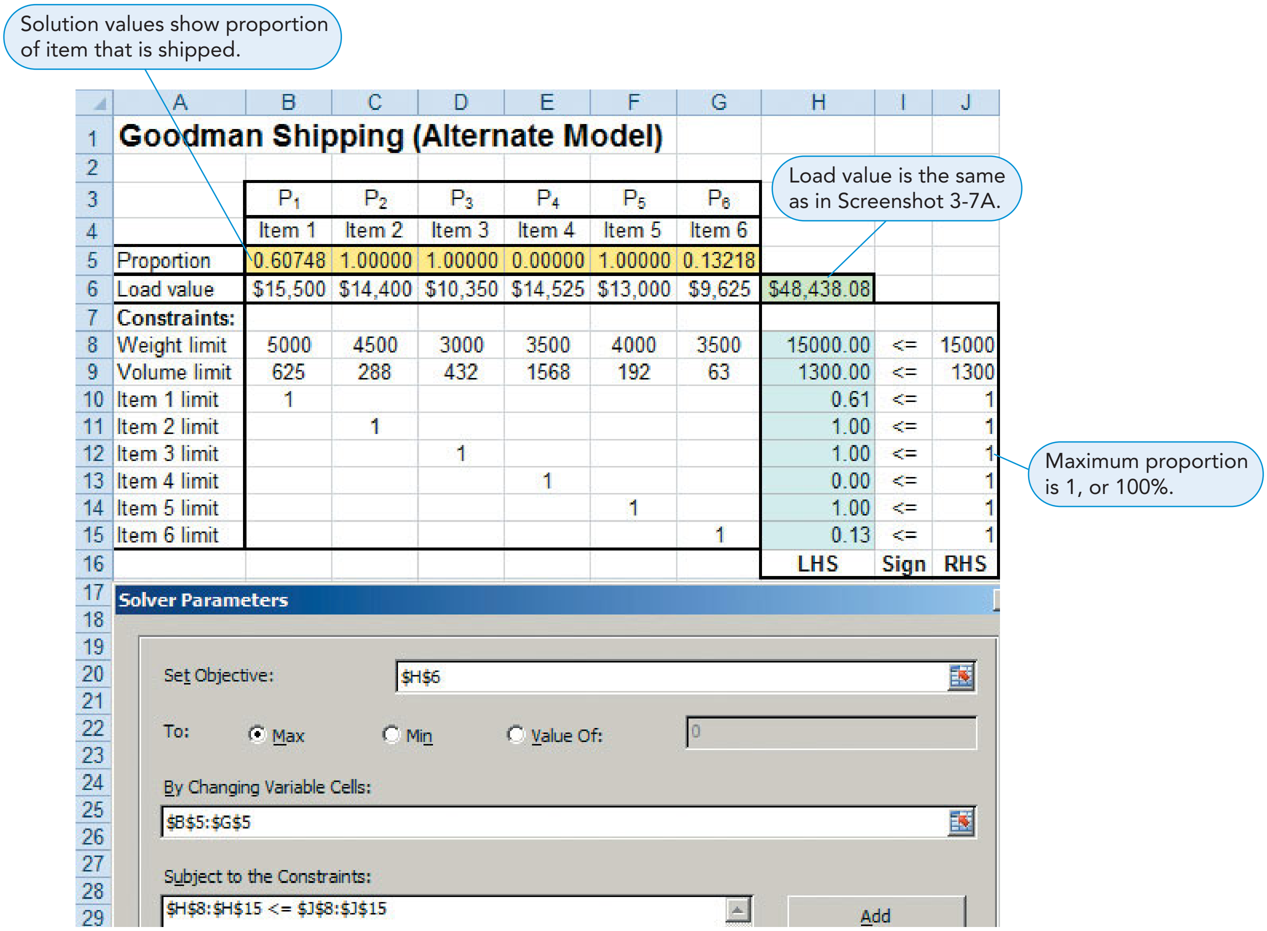
Expanded Vehicle Loading Problem—Allocation Problem

In the previous example, Goodman had only a single truck and needed to load all items on the same truck. Let us now assume that Goodman has the option of replacing his single truck (with a weight capacity of 15,000 pounds and a volume capacity of 1,300 cubic feet) with two smaller trucks (each with a weight capacity of 10,000 pounds and a volume capacity of 900 cubic feet). Item availabilities and other data are still as shown in Table 3.7. If he uses two trucks, Goodman wants to ensure that they are loaded in an equitable manner. That is, the same total weight should be loaded on both trucks. Total volumes loaded in the two trucks can, however, be different. If the fixed cost of operating the two smaller trucks is $5,000 more than the current cost of operating just a single truck, should Goodman go with the two trucks?

Allocation problems involve deciding how much of an item to allocate to the different choices that are available.

In this revised model, Goodman has to decide how to allocate the six items between the two trucks. Note that it is possible for the total quantity of an item to be split between the two trucks. That is, we can load a portion of an item in the first truck and load part or all of the remaining

SCREENSHOT 3-7B Excel Layout and Solver Entries for Goodman Shipping—Alternate Model



portion in the other truck. Because the decision involves an allocation (of items to trucks, in Goodman's case), this type of problem is called an *allocation* problem.

Using double-subscripted variables is a convenient way of formulating many LP models.

FORMULATING THE PROBLEM The decision variables here need to specify how much of each item should be loaded on each truck. Let the double-subscripted variable W_{i1} be the weight (in pounds) of each item i loaded on the first truck and W_{i2} be the weight (in pounds) of each item i loaded on the second truck. The LP model for this expanded vehicle loading problem can then be formulated as follows:

$$\begin{aligned}\text{Maximize load value} = {} & \$3.10(W_{11} + W_{12}) + \$3.20(W_{21} + W_{22}) + \$3.45(W_{31} + W_{32}) \\ & + \$4.15(W_{41} + W_{42}) + \$3.25(W_{51} + W_{52}) + \$2.75(W_{61} + W_{62})\end{aligned}$$

subject to the constraints

$$\begin{aligned}
& W_{11} + W_{21} + W_{31} + W_{41} + W_{51} + W_{61} && \leq 10{,}000 && \text{(weight limit of truck 1)} \\
& 0.125\,W_{11} + 0.064\,W_{21} + 0.144\,W_{31} + \\
& \quad 0.448\,W_{41} + 0.048\,W_{51} + 0.018\,W_{61} && \leq 900 && \text{(volume limit of truck 1)} \\
& W_{12} + W_{22} + W_{32} + W_{42} + W_{52} + W_{62} && \leq 10{,}000 && \text{(weight limit of truck 2)} \\
& 0.125\,W_{12} + 0.064\,W_{22} + 0.144\,W_{32} + \\
& \quad 0.448\,W_{42} + 0.048\,W_{52} + 0.018\,W_{62} && \leq 900 && \text{(volume limit of truck 2)} \\
& W_{11} + W_{12} && \leq 5{,}000 && \text{(item 1 availability)} \\
& W_{21} + W_{22} && \leq 4{,}500 && \text{(item 2 availability)} \\
& W_{31} + W_{32} && \leq 3{,}000 && \text{(item 3 availability)} \\
& W_{41} + W_{42} && \leq 3{,}500 && \text{(item 4 availability)}
\end{aligned}$$

$$W_{51} + W_{52} \leq 4{,}000 \quad \text{(item 5 availability)}$$
$$W_{61} + W_{62} \leq 3{,}500 \quad \text{(item 6 availability)}$$
$$W_{11} + W_{21} + W_{31} + W_{41} + W_{51} + W_{61} = W_{12} + W_{22} + W_{32} + W_{42} + W_{52} + W_{62}$$
$$\text{(same weight in both trucks)}$$
$$\text{All variables} \geq 0 \quad \text{(nonnegativity)}$$

File: 3-8.xls

SOLVING THE PROBLEM Screenshot 3-8 shows the Excel layout and Solver entries for Goodman Shipping's allocation LP model. For the constraint that ensures the same total weight is loaded on both trucks, the Excel layout includes formulas for both the LHS (cell N18) and RHS (cell P18) entries. While the formula in cell N18 uses the usual SUMPRODUCT function, the formula in cell P18 is

Cell P18: =SUM(H5:M5)

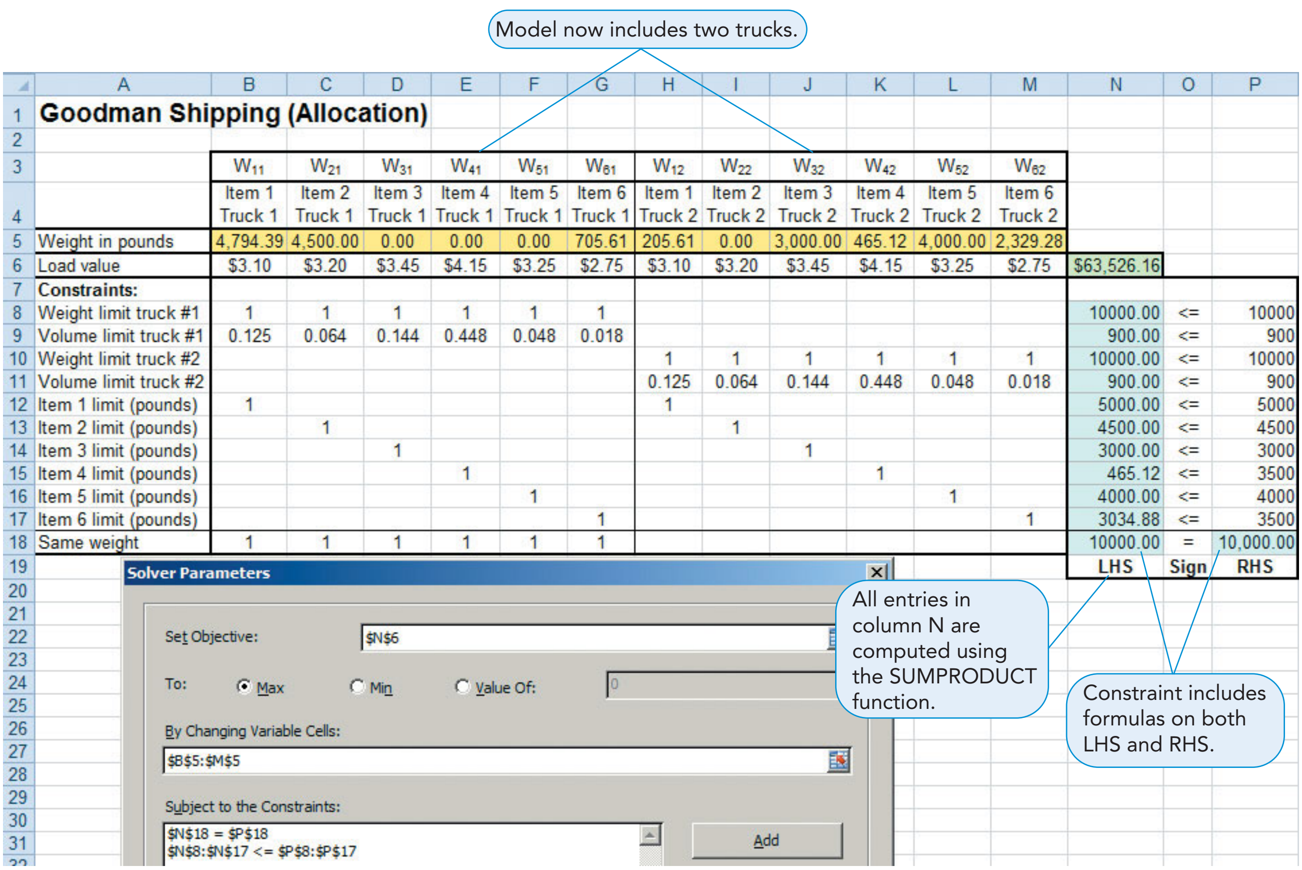
INTERPRETING THE RESULTS Screenshot 3-8 indicates that the optimal solution to Goodman Shipping's allocation problem yields a total value of $63,526.16. This is an increase of $15,088.08 over the load value of $48,438.08 realizable with just the single truck. Because this more than compensates for the increased $5,000 operating cost, Goodman should replace his single truck with the two smaller ones. Both trucks are fully loaded from both weight and volume perspectives. All available quantities of items 1, 2, 3, and 5 are loaded. Most of item 6 is loaded (3,034.88 of 3,500 pounds available), while only about 13.29% (465.12 of 3,500 pounds available) of item 4 is loaded.

Transportation Problem

A problem involving transporting goods from several origins to several destinations efficiently is called a transportation problem.

A transportation, or shipping, problem involves determining the amount of goods or number of items to be transported from a number of origins (or supply locations) to a number of destinations (or demand locations). The objective usually is to minimize total shipping costs or

SCREENSHOT 3-8 Excel Layout and Solver Entries for Goodman Shipping—Allocation



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	Goodman Shipping (Allocation)															
2																
3		W_{11}	W_{21}	W_{31}	W_{41}	W_{51}	W_{61}	W_{12}	W_{22}	W_{32}	W_{42}	W_{52}	W_{62}			
4		Item 1 Truck 1	Item 2 Truck 1	Item 3 Truck 1	Item 4 Truck 1	Item 5 Truck 1	Item 6 Truck 1	Item 1 Truck 2	Item 2 Truck 2	Item 3 Truck 2	Item 4 Truck 2	Item 5 Truck 2	Item 6 Truck 2			
5	Weight in pounds	4,794.39	4,500.00	0.00	0.00	0.00	705.61	205.61	0.00	3,000.00	465.12	4,000.00	2,329.28			
6	Load value	$3.10	$3.20	$3.45	$4.15	$3.25	$2.75	$3.10	$3.20	$3.45	$4.15	$3.25	$2.75	$63,526.16		
7	Constraints:															
8	Weight limit truck #1	1	1	1	1	1	1							10000.00	<=	10000
9	Volume limit truck #1	0.125	0.064	0.144	0.448	0.048	0.018							900.00	<=	900
10	Weight limit truck #2							1	1	1	1	1	1	10000.00	<=	10000
11	Volume limit truck #2							0.125	0.064	0.144	0.448	0.048	0.018	900.00	<=	900
12	Item 1 limit (pounds)	1						1						5000.00	<=	5000
13	Item 2 limit (pounds)		1						1					4500.00	<=	4500
14	Item 3 limit (pounds)			1						1				3000.00	<=	3000
15	Item 4 limit (pounds)				1						1			465.12	<=	3500
16	Item 5 limit (pounds)					1						1		4000.00	<=	4000
17	Item 6 limit (pounds)						1						1	3034.88	<=	3500
18	Same weight	1	1	1	1	1	1							10000.00	=	10,000.00
19														LHS	Sign	RHS

IN ACTION Swift & Company Uses Linear Programming to Schedule Operations at Five Plants

Swift & Company, a privately held Colorado based protein-processing company, has three business segments: Swift Beef, Swift Pork, and Swift Australia. Beef and related products constitute the largest portion of the company's annual sales of over $8 billion. Swift operates five plants with each plant capable of processing approximately 18,000 to 25,000 head of cattle per plant per week, which translates to over 6 billion pounds of beef delivered annually.

Because beef is highly perishable, Swift's customers specify a 10 to 14 day maximum age for the beef upon delivery. Production schedulers must therefore be aware of inventory quantities and age. Tight margins means that optimizing cattle procurement and product mix is essential to improving profitability. With over 1,500 stock keeping units, 30,000 shipping locations, and the high costs and many sources of variability in raw material, Swift's optimization problem is both complex and difficult.

Working in cooperation with Swift, analysts at Aspen Technology developed an integrated system of 45 linear programming models that address Swift's four critical needs: (1) provide near real time information on product availability, (2) accurately control inventories, (3) provide the ability to sell unsold production with maximum margins, and (4) provide the ability to reoptimize the use of raw material to satisfy changes in demand. The scheduling models produce shift-level and daily schedules for each plant over a 28-day planning horizon. Swift realized total audited benefits of $12.74 million in just the first year after this new system was implemented.

Source: Based on A. Bixby, B. Downs, and M. Self. "A Scheduling and Capable-to-Promise Application for Swift & Company," *Interfaces*, 36, 1 (January–February 2006): 69–86.

distances. Constraints in this type of problem deal with capacities or supplies at each origin and requirements or demands at each destination.

Like the assignment problem, the transportation problem is also an example of a network flow problem. We will study this type of problem in greater detail in Chapter 5.

3.7 Blending Applications

Diet Problem

The diet problem, one of the earliest applications of LP, was originally used by hospitals to determine the most economical diet for patients. Known in agricultural applications as the feed mix problem, the diet problem involves specifying a food or feed ingredient combination that satisfies stated nutritional requirements at a minimum cost level. An example follows.

The Whole Food Nutrition Center uses three different types of bulk grains to blend a natural breakfast cereal that it sells by the pound. The store advertises that each 2-ounce serving of the cereal, when taken with $^1/_2$ cup of whole milk, meets an average adult's minimum daily requirement for protein, riboflavin, phosphorus, and magnesium. The cost of each bulk grain and the protein, riboflavin, phosphorus, and magnesium units per pound of each are shown in Table 3.8.

The minimum adult daily requirement (called the U.S. Recommended Daily Allowance [USRDA]) for protein is 3 units, for riboflavin is 2 units, for phosphorus is 1 unit, and for magnesium is 0.425 units. Whole Food wants to select the blend of grains that will meet the USRDA at a minimum cost.

In blending applications, the decision variables typically denote the amount of each ingredient that should be used to make the product(s).

FORMULATING AND SOLVING THE PROBLEM The decision variables in blending applications typically define the amount of each ingredient that should be used to make the product(s). It is interesting to contrast this with product mix problems, where the decision variables define the

TABLE 3.8 Requirements for Whole Food's Natural Cereal

GRAIN	COST PER POUND (CENTS)	PROTEIN (UNITS/LB)	RIBOFLAVIN (UNITS/LB)	PHOSPHORUS (UNITS/LB)	MAGNESIUM (UNITS/LB)
A	33	22	16	8	5
B	47	28	14	7	0
C	38	21	25	9	6

number of units to make of each product. That is, blending problems typically make decisions regarding amounts of each *input* (resource) to use, while product mix problems make decisions regarding numbers of each *output* to make.

In Whole Food Nutrition's case, the ingredients (inputs) are the three different types of bulk grains. We let

$$A = \text{pounds of grain A to use in one 2-ounce serving of cereal}$$
$$B = \text{pounds of grain B to use in one 2-ounce serving of cereal}$$
$$C = \text{pounds of grain C to use in one 2-ounce serving of cereal}$$

This is the objective function:

$$\text{Minimize total cost of mixing a 2-ounce serving of cereal} = \$0.33A + \$0.47B + \$0.38C$$

subject to the constraints

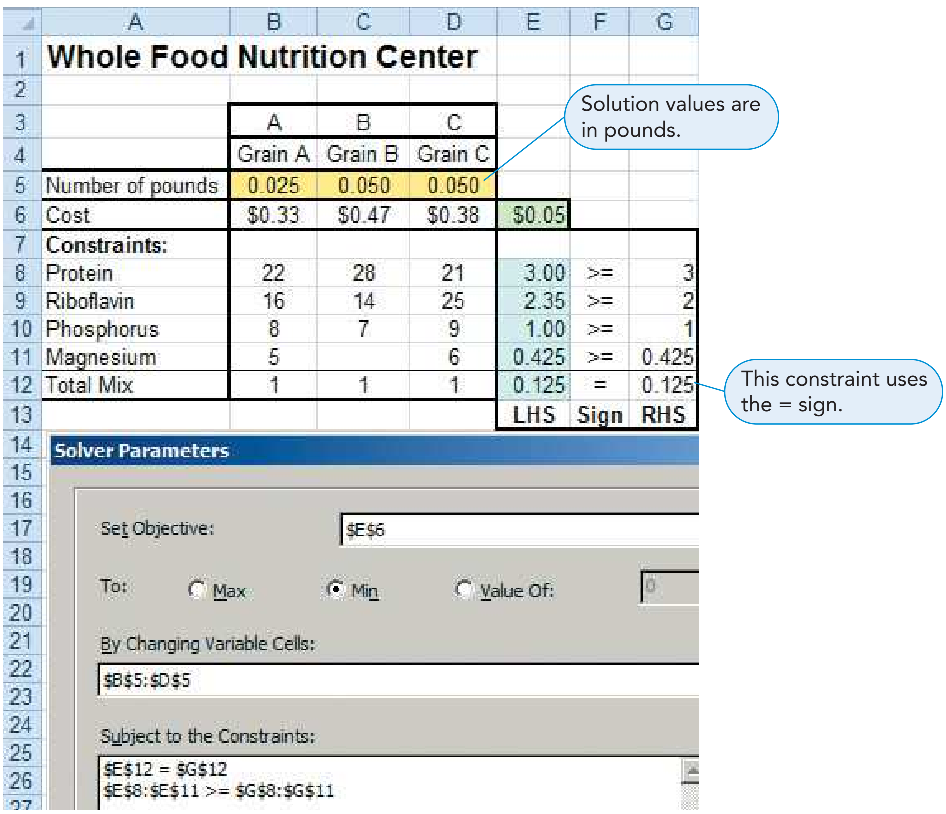
$$\begin{array}{lll} 22A + 28B + 21C & \geq 3 & \text{(protein units)} \\ 16A + 14B + 25C & \geq 2 & \text{(riboflavin units)} \\ 8A + 7B + 9C & \geq 1 & \text{(phosphorus units)} \\ 5A + 6C & \geq 0.425 & \text{(magnesium units)} \\ A + B + C & = 0.125 & \text{(total mix is 2 ounces, or 0.125 pound)} \\ A, B, C & \geq 0 & \end{array}$$

Screenshot 3-9 shows the Excel layout and Solver entries for this LP model.

File: 3-9.xls

INTERPRETING THE RESULTS The solution to Whole Food Nutrition Center's problem requires mixing together 0.025 pounds of grain A, 0.050 pounds of grain B, and 0.050 pounds of grain C. Another way of stating this solution is in terms of a 2-ounce serving of each grain: 0.4 ounces of grain A, 0.8 ounces of grain B, and 0.8 ounces of grain C in each 2-ounce serving of cereal. The cost per serving is $0.05.

SCREENSHOT 3-9
Excel Layout and Solver Entries for Whole Food Nutrition Center



	A	B	C	D	E	F	G
1	Whole Food Nutrition Center						
2							
3		A	B	C			
4		Grain A	Grain B	Grain C			
5	Number of pounds	0.025	0.050	0.050			
6	Cost	$0.33	$0.47	$0.38	$0.05		
7	Constraints:						
8	Protein	22	28	21	3.00	>=	3
9	Riboflavin	16	14	25	2.35	>=	2
10	Phosphorus	8	7	9	1.00	>=	1
11	Magnesium	5		6	0.425	>=	0.425
12	Total Mix	1	1	1	0.125	=	0.125
13					LHS	Sign	RHS

TABLE 3.9 Ingredient Data for Low Knock Oil

CRUDE OIL TYPE	COMPOUND A (%)	COMPOUND B (%)	COMPOUND C (%)	COST/BARREL ($)	AVAIL. (BARRELS)
$X100$	35	25	35	86	15,000
$X200$	50	30	15	92	32,000
$X300$	60	20	15	95	24,000

Blending Problem

In most practical blending problems, the ingredients are usually used to make more than one product.

In the preceding diet problem, the ingredients (grains) had to be mixed to create just a single product (cereal). Diet and feed mix problems are actually special cases of a more general class of LP problems known as *blending problems*. Blending problems are very common in chemical industries, and they arise when a decision must be made regarding the blending of two or more ingredients (or resources) to produce two or more products (or end items). The ingredients must be blended in such a manner that each final product satisfies specific requirements regarding its composition. In addition, ingredients may have limitations regarding their availabilities, and products may have conditions regarding their demand. The following example deals with an application frequently seen in the petroleum industry: the blending of crude oils to produce different grades of gasoline.

Major oil refineries use LP for blending crude oils to produce gasoline grades.

The Low Knock Oil Company produces three grades of gasoline for industrial distribution. The three grades—premium, regular, and economy—are produced by refining a blend of three types of crude oil: type $X100$, type $X200$, and type $X300$. Each crude oil differs not only in cost per barrel but in its composition as well. Table 3.9 indicates the percentage of three crucial compounds found in each of the crude oils, the cost per barrel for each, and the maximum weekly availability of each.

Table 3.10 indicates the weekly demand for each grade of gasoline and the specific conditions on the amounts of the different compounds that each grade of gasoline should contain. The table shows, for example, that in order for gasoline to be classified as premium grade, it must contain at least 55% compound A. Low Knock's management must decide how many barrels of each type of crude oil to buy each week for blending to satisfy demand at minimum cost.

FORMULATING THE PROBLEM As noted in the diet problem, the decision variables in blending applications typically denote the amount of each ingredient that should be used to make the product(s). In Low Knock's case, the ingredients are the crude oils, and the products are the grades of gasoline. Because there are three ingredients that are blended to create three products, we need a total of 9 $(= 3 \times 3)$ decision variables. We let

P_1 = barrels of $X100$ crude blended to produce the premium grade
P_2 = barrels of $X200$ crude blended to produce the premium grade
P_3 = barrels of $X300$ crude blended to produce the premium grade
R_1 = barrels of $X100$ crude blended to produce the regular grade
R_2 = barrels of $X200$ crude blended to produce the regular grade
R_3 = barrels of $X300$ crude blended to produce the regular grade
E_1 = barrels of $X100$ crude blended to produce the economy grade
E_2 = barrels of $X200$ crude blended to produce the economy grade
E_3 = barrels of $X300$ crude blended to produce the economy grade

TABLE 3.10 Gasoline Data for Low Knock Oil

GASOLINE TYPE	COMPOUND A	COMPOUND B	COMPOUND C	DEMAND (BARRELS)
Premium	$\geq 55\%$	$\leq 23\%$		14,000
Regular		$\geq 25\%$	$\leq 35\%$	22,000
Economy	$\geq 40\%$		$\leq 25\%$	25,000

We can calculate the total amount produced of each gasoline grade by adding the amounts of the three crude oils used to create that grade. For example,

$$\text{Total amount of premium grade gasoline produced} = P_1 + P_2 + P_3$$

Likewise, we can calculate the total amount used of each crude oil type by adding the amounts of that crude oil used to create the three gasoline grades. For example,

$$\text{Total amount of crude oil type } X100 \text{ used} = P_1 + R_1 + E_1$$

The objective is to minimize the total cost of the crude oils used and can be written as

$$\text{Minimize total cost} = \$86(P_1 + R_1 + E_1) + \$92(P_2 + R_2 + E_2) + \$95(P_3 + R_3 + E_3)$$

subject to the constraints

$$\begin{aligned}
P_1 + R_1 + E_1 &\le 15{,}000 && \text{(availability of } X100 \text{ crude oil)}\\
P_2 + R_2 + E_2 &\le 32{,}000 && \text{(availability of } X200 \text{ crude oil)}\\
P_3 + R_3 + E_3 &\le 24{,}000 && \text{(availability of } X300 \text{ crude oil)}\\
P_1 + P_2 + P_3 &\ge 14{,}000 && \text{(demand for premium gasoline)}\\
R_1 + R_2 + R_3 &\ge 22{,}000 && \text{(demand for regular gasoline)}\\
E_1 + E_2 + E_3 &\ge 25{,}000 && \text{(demand for economy gasoline)}
\end{aligned}$$

Observe that we have written the demand constraints as $\ge$ conditions in this problem. The reason is that the objective function in this problem is to minimize the total cost. If we write the demand constraints also as $\le$ conditions, the optimal solution could result in a trivial "don't make anything, don't spend anything" situation.[1] As we have seen in several examples before now, if the objective function is a minimization function, there must be at least one constraint in the problem that forces the optimal solution away from the origin (i.e., zero values for all decision variables).

Blending constraints specify the compositions of each product.

Next, we come to the blending constraints that specify the amounts of the different compounds that each grade of gasoline can contain. First, we know that at least 55% of each barrel of premium gasoline must be compound A. To write this constraint, we note that

$$\text{Amount of compound A in premium grade gasoline} = 0.35P_1 + 0.50P_2 + 0.60P_3$$

If we divide this amount by the total amount of premium grade gasoline produced $(= P_1 + P_2 + P_3)$, we get the total portion of compound A in this grade of gasoline. Therefore, the constraint may be written as

$$(0.35P_1 + 0.50P_2 + 0.60P_3)/(P_1 + P_2 + P_3) \ge 0.55 \qquad \text{(compound A in premium grade)}$$

The other compound specifications may be written in a similar fashion, as follows:

$$\begin{aligned}
(0.25P_1 + 0.30P_2 + 0.20P_3)/(P_1 + P_2 + P_3) &\le 0.23 && \text{(compound B in premium grade)}\\
(0.25R_1 + 0.30R_2 + 0.20R_3)/(R_1 + R_2 + R_3) &\ge 0.25 && \text{(compound B in regular grade)}\\
(0.35R_1 + 0.15R_2 + 0.15R_3)/(R_1 + R_2 + R_3) &\le 0.35 && \text{(compound C in regular grade)}\\
(0.35E_1 + 0.50E_2 + 0.60E_3)/(E_1 + E_2 + E_3) &\ge 0.40 && \text{(compound A in economy grade)}\\
(0.35E_1 + 0.15E_2 + 0.15E_3)/(E_1 + E_2 + E_3) &\le 0.25 && \text{(compound C in economy grade)}
\end{aligned}$$

Finally, we have the nonnegativity constraints:

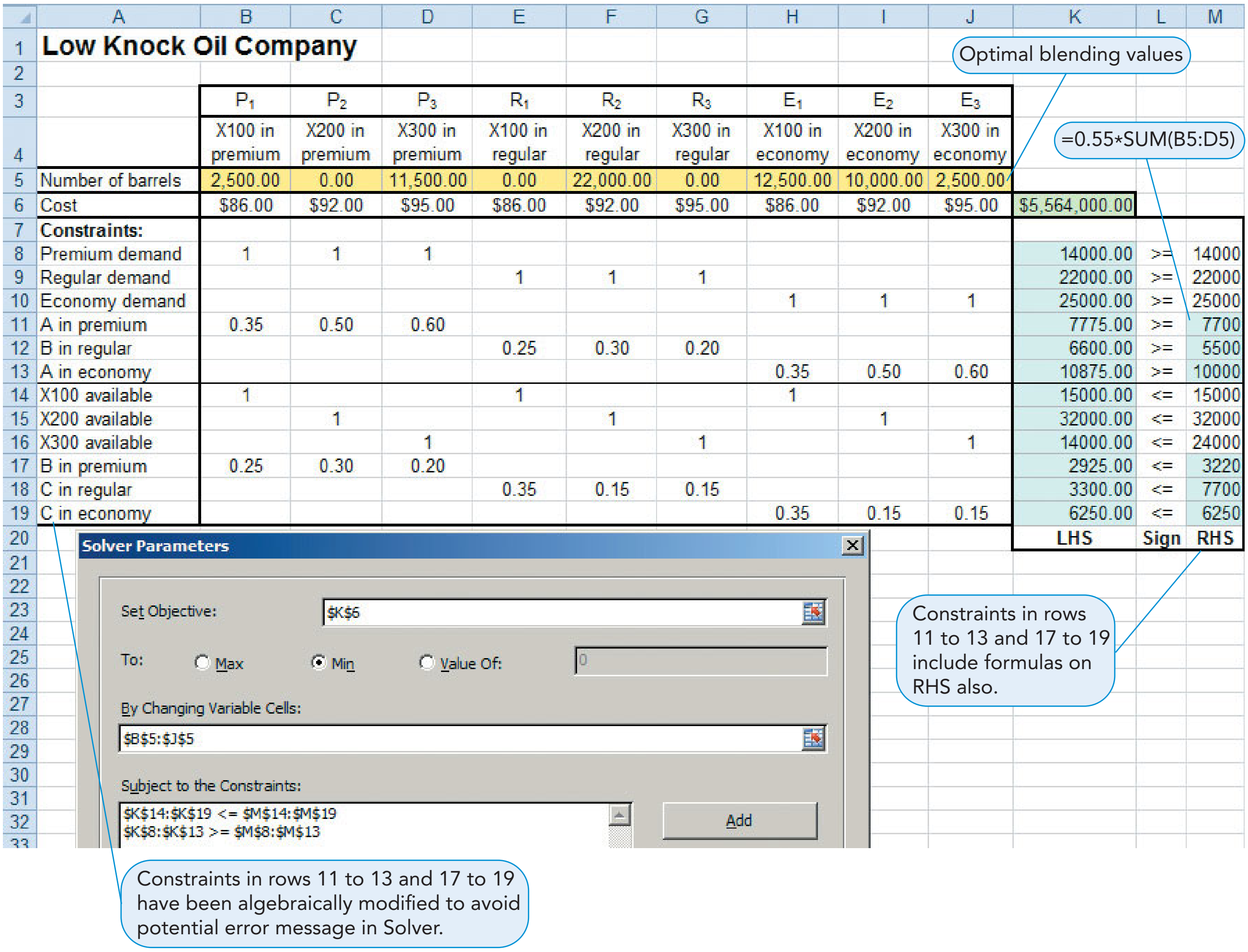
$$P_1, P_2, P_3, R_1, R_2, R_3, E_1, E_2, E_3 \ge 0 \qquad \text{(nonnegativity)}$$

WWW
File: 3-10.xls

SOLVING THE PROBLEM Screenshot 3-10 shows the Excel layout and Solver entries for this LP model. As in the portfolio selection problem, to avoid potential error messages from Solver (see the *Excel Notes* on page 79), we have algebraically modified all the compound specification

[1] This, of course, assumes that the total availability of the three crude oils is sufficient to satisfy the total demand for the three gasoline grades. If total availability is less than total demand, we need to write the demand constraints as $\le$ expressions and the availability constraints as $\ge$ expressions.

SCREENSHOT 3-10 **Excel Layout and Solver Entries for Low Knock Oil**



	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Low Knock Oil Company**												
2													
3		P_1	P_2	P_3	R_1	R_2	R_3	E_1	E_2	E_3			
4		X100 in premium	X200 in premium	X300 in premium	X100 in regular	X200 in regular	X300 in regular	X100 in economy	X200 in economy	X300 in economy			
5	Number of barrels	2,500.00	0.00	11,500.00	0.00	22,000.00	0.00	12,500.00	10,000.00	2,500.00			
6	Cost	$86.00	$92.00	$95.00	$86.00	$92.00	$95.00	$86.00	$92.00	$95.00	$5,564,000.00		
7	**Constraints:**												
8	Premium demand	1	1	1							14000.00	>=	14000
9	Regular demand				1	1	1				22000.00	>=	22000
10	Economy demand							1	1	1	25000.00	>=	25000
11	A in premium	0.35	0.50	0.60							7775.00	>=	7700
12	B in regular				0.25	0.30	0.20				6600.00	>=	5500
13	A in economy							0.35	0.50	0.60	10875.00	>=	10000
14	X100 available	1			1			1			15000.00	<=	15000
15	X200 available		1			1			1		32000.00	<=	32000
16	X300 available			1			1			1	14000.00	<=	24000
17	B in premium	0.25	0.30	0.20							2925.00	<=	3220
18	C in regular				0.35	0.15	0.15				3300.00	<=	7700
19	C in economy							0.35	0.15	0.15	6250.00	<=	6250
20											**LHS**	**Sign**	**RHS**

constraints in our Excel implementation of this model. For example, the constraint specifying the portion of compound A in premium grade gasoline is modified as

$$0.35P_1 + 0.50P_2 + 0.60P_3 \geq 0.55(P_1 + P_2 + P_3)$$

The six compound specification constraints therefore have Excel formulas in both the LHS cells (K11:K13, K17:K19) and RHS cells (M11:M13, M17:M19). The LHS cells use the usual SUMPRODUCT function.

INTERPRETING THE RESULTS Screenshot 3-10 indicates that the blending strategy will cost Low Knock Oil \$5,564,000 and require it to mix the three types of crude oil as follows:

P_1 = 2,500 barrels of $X100$ crude oil to make premium grade gasoline
P_3 = 11,500 barrels of $X300$ crude oil to make premium grade gasoline
R_2 = 22,000 barrels of $X200$ crude oil to make regular grade gasoline
E_1 = 12,500 barrels of $X100$ crude oil to make economy grade gasoline
E_2 = 10,000 barrels of $X200$ crude oil to make economy grade gasoline
E_3 = 2,500 barrels of $X300$ crude oil to make economy grade gasoline

The entire demand for the three gasoline grades is met. Low Knock should buy all available barrels of crude oil types $X100$ and $X200$ and only 14,000 barrels of the 24,000 barrels available of type $X300$. (Note: This model has multiple optimal solutions.)

3.8 Multiperiod Applications

Multiperiod problems are perhaps the most challenging application of LP.

Perhaps the most challenging application of LP is in modeling multiperiod scenarios. These are situations in which the decision maker has to determine the optimal decisions for several periods (e.g., weeks, months). What makes these problems especially difficult is that the decision choices in later periods are directly dependent on the decisions made in earlier periods. We discuss two examples in the following sections to illustrate this feature. The first example deals with a multiperiod production scheduling problem. The second example involves the establishment of a multiperiod financial sinking fund.

Production Scheduling Problem

Models for multiperiod production planning typically have to include several, often conflicting, factors.

Setting a low-cost production schedule over a period of weeks or months is a difficult and important management problem in most plants. Because most companies produce more than one product, the scheduling process is often quite complex. The production manager has to consider several factors, such as labor capacity, inventory and storage costs, space limitations, product demand, and labor relations. These factors often conflict with each other. For example, it is desirable to produce the same number of each product each period in order to simplify planning and scheduling of workers and machines. However, the need to keep inventory carrying costs down suggests producing in each period only what is needed that period. As we shall see in the following problem, LP is an effective tool for resolving such conflicts and identifying a production schedule that will minimize the total cost of production and inventory holding. Production scheduling is especially amenable to solution by LP because it is a problem that must be solved on a regular basis. When the objective function and constraints for a firm are established, the inputs can easily be changed each period to provide an updated schedule.

Production decisions in later periods are directly dependent on decisions made in earlier periods.

Basically, a multiperiod problem resembles the product mix model for each period in the planning horizon, with the additional issue of inventory from one period to the next to be considered. The objective is to either maximize profit or to minimize the total cost (production plus inventory) of carrying out the task. As noted earlier, production decision choices in later periods are directly affected by decisions made in earlier periods. An example follows.

Greenberg Motors, Inc., manufactures two different electrical motors for sale under contract to Drexel Corp., a well-known producer of small kitchen appliances. Its model GM3A is found in many Drexel food processors, and its model GM3B is used in the assembly of blenders.

Three times each year, the procurement officer at Drexel contracts Irwin Greenberg, the founder of Greenberg Motors, to place a monthly order for each of the coming four months. Drexel's demand for motors varies each month, based on its own sales forecasts, production capacity, and financial position. Greenberg has just received the January–April order and must begin his own four-month production plan. The demand for motors is shown in Table 3.11.

The following additional data are available regarding Greenberg's problem:

1. Production costs are currently \$10 per GM3A motor produced and \$6 per GM3B unit. However, a planned wage increase going into effect on March 1 will raise each figure by 10%.
2. Each GM3A motor held in stock costs \$0.18 per month, and each GM3B has a holding cost of \$0.13 per month. Greenberg's accountants allow monthly ending inventories as an acceptable approximation to the average inventory levels during the month.
3. Greenberg is starting the new four-month production cycle with a change in design specifications that has left no old motors of either type in stock on January 1.
4. Greenberg wants to have ending inventories of 450 GM3As and 300 GM3Bs at the end of April.
5. The storage area can hold a maximum of 3,300 motors of either type (they are similar in size) at any one time. Additional storage space is very expensive and is therefore not available as an option.

TABLE 3.11 Four-Month Order Schedule for Greenberg Motors

MODEL	JANUARY	FEBRUARY	MARCH	APRIL
GM3A	800	700	1,000	1,100
GM3B	1,000	1,200	1,400	1,400

6. Greenberg has a no-layoff policy, which has been effective in preventing unionization of the shop. The company has a base employment level of 2,240 labor hours per month, and, by contract, this level of labor must be used each month. In busy periods, however, the company has the option of bringing on board two skilled former employees who are now retired. Each of these employees can provide up to 160 labor hours per month.
7. Each GM3A motor produced requires 1.3 hours of labor, and each GM3B takes a worker 0.9 hours to assemble.

Double-subscripted variables are very convenient in formulating multiperiod problems.

FORMULATING THE PROBLEM Just as in the product mix problem, the primary decision variables here define the number of units of each product (motors) to make. However, because production of motors occurs in four separate months, we need to define decision variables to define the production of each motor in each month. Using double-subscripted variables is a convenient way of defining the decision variables in this LP model. We let

P_{At} = number of model GM3A motors produced in month t ($t = 1, 2, 3, 4$ for January–April)

P_{Bt} = number of model GM3B motors produced in month t

Using these variables, the total production cost may be written as follows (recall that unit costs go up 10% in March):

$$\text{Cost of production} = \$10P_{A1} + \$10P_{A2} + \$11P_{A3} + \$11P_{A4} + \$6P_{B1} + \$6P_{B2} + \$6.60P_{B3} + \$6.60P_{B4}$$

To keep track of the inventory carried over from one month to the next, we introduce a second set of decision variables. Let

I_{At} = level of on-hand inventory for GM3A motors at end of month t ($t = 1, 2, 3, 4$)

I_{Bt} = level of on-hand inventory for GM3B motors at end of month t ($t = 1, 2, 3, 4$)

Using these variables, the total inventory carrying costs may be written as

$$\text{Cost of carrying inventory} = \$0.18I_{A1} + 0.18I_{A2} + 0.18I_{A3} + 0.18I_{A4} + 0.13I_{B1} + 0.13I_{B2} + 0.13I_{B3} + 0.13I_{B4}$$

The objective function is then

$$\begin{aligned}\text{Minimize total costs} &= \text{cost of production} + \text{cost of carrying inventory}\\ &= 10P_{A1} + 10P_{A2} + 11P_{A3} + 11P_{A4} + 6P_{B1} + 6P_{B2} + 6.60P_{B3}\\ &\quad + 6.60P_{B4} + 0.18I_{A1} + 0.18I_{A2} + 0.18I_{A3} + 0.18I_{A4} + 0.13I_{B1}\\ &\quad + 0.13I_{B2} + 0.13I_{B3} + 0.13I_{B4}\end{aligned}$$

Balance constraints specify the relationship between the previous period's closing inventory, this period's production, this period's sales, and this period's closing inventory.

In all multiperiod problems, we need to write a *balance equation*, or *constraint*, for each product for each period. Each balance equation specifies the relationship between the previous period's ending inventory, the current period's production, the current period's sales, and the current period's ending inventory. Specifically, the balance equation states that the inventory at the end of the current period is given by

$$\begin{pmatrix}\text{inventory}\\ \text{at end}\\ \text{of previous}\\ \text{period}\end{pmatrix} + \begin{pmatrix}\text{current}\\ \text{period's}\\ \text{production}\end{pmatrix} - \begin{pmatrix}\text{current}\\ \text{period's}\\ \text{sales}\end{pmatrix} = \begin{pmatrix}\text{inventory}\\ \text{at end}\\ \text{of current}\\ \text{period}\end{pmatrix}$$

In Greenberg's case, we are starting with no old motors in stock on January 1. Recalling that January's demand for GM3As is 800 and for GM3Bs is 1,000, we can write the balance constraints for January as

$$0 + P_{A1} - 800 = I_{A1} \qquad \text{(GM3A motors in January)}$$
$$0 + P_{B1} - 1{,}000 = I_{B1} \qquad \text{(GM3B motors in January)}$$

In a similar fashion, the balance constraints for February, March, and April may be written as follows:

$$I_{A1} + P_{A2} - 700 = I_{A2} \qquad \text{(GM3A motors in February)}$$
$$I_{B1} + P_{B2} - 1{,}200 = I_{B2} \qquad \text{(GM3B motors in February)}$$

$$I_{A2} + P_{A3} - 1{,}000 = I_{A3} \qquad \text{(GM3A motors in March)}$$
$$I_{B2} + P_{B3} - 1{,}400 = I_{B3} \qquad \text{(GM3B motors in March)}$$
$$I_{A3} + P_{A4} - 1{,}100 = I_{A4} \qquad \text{(GM3A motors in April)}$$
$$I_{B3} + P_{B4} - 1{,}400 = I_{B4} \qquad \text{(GM3B motors in April)}$$

If Greenberg wants to have ending inventories of 450 GM3As and 300 GM3Bs at the end of April, we add the constraints

$$I_{A4} = 450$$
$$I_{B4} = 300$$

Although the balance constraints address demand, they do not consider storage space or labor requirements. First, we note that the storage area for Greenberg Motors can hold a maximum of 3,300 motors of either type. Therefore, we write

$$I_{A1} + I_{B1} \leq 3{,}300$$
$$I_{A2} + I_{B2} \leq 3{,}300$$
$$I_{A3} + I_{B3} \leq 3{,}300$$
$$I_{A4} + I_{B4} \leq 3{,}300$$

Employment constraints are specified for each period.

Second, we note that Greenberg must use at least 2,240 labor hours per month and could potentially have up to 2,560 labor hours ($= 2{,}240 + 160 \times 2$) per month. Because each GM3A motor produced requires 1.3 hours of labor and each GM3B takes a worker 0.9 hours to assemble, we write the labor constraints as

$$1.3P_{A1} + 0.9P_{B1} \geq 2{,}240 \qquad \text{(January labor minimum)}$$
$$1.3P_{A1} + 0.9P_{B1} \leq 2{,}560 \qquad \text{(January labor maximum)}$$
$$1.3P_{A2} + 0.9P_{B2} \geq 2{,}240 \qquad \text{(February labor minimum)}$$
$$1.3P_{A2} + 0.9P_{B2} \leq 2{,}560 \qquad \text{(February labor maximum)}$$
$$1.3P_{A3} + 0.9P_{B3} \geq 2{,}240 \qquad \text{(March labor minimum)}$$
$$1.3P_{A3} + 0.9P_{B3} \leq 2{,}560 \qquad \text{(March labor maximum)}$$
$$1.3P_{A4} + 0.9P_{B4} \geq 2{,}240 \qquad \text{(April labor minimum)}$$
$$1.3P_{A4} + 0.9P_{B4} \leq 2{,}560 \qquad \text{(April labor maximum)}$$

Finally, we have the nonnegativity constraints:

$$\text{All variables} \geq 0$$

File: 3-11.xls, sheet: 3-11A

SOLVING THE PROBLEM There are several ways of setting up the Greenberg Motors problem in Excel. The setup shown in Screenshot 3-11A follows the usual logic we have used in all problems so far; that is, all parameters associated with a specific decision variable are modeled in the same column.

In setting up the balance constraints in Screenshot 3-11A, we have algebraically modified each equation by moving all variables to the LHS of the equation and the constants to the RHS. This is a convenient way of implementing these constraints. For example, the balance constraint for GM3A motors in February, which currently reads

$$I_{A1} + P_{A2} - 700 = I_{A2}$$

is modified as

$$I_{A1} + P_{A2} - I_{A2} = 700$$

INTERPRETING THE RESULTS The solution to Greenberg's problem, summarized in Table 3.12, indicates that the four-month total cost is \$76,301.62. The solution requires Greenberg to use the two former employees to their maximum extent (160 hours each; 320 hours total) in three of the four months and for 115 hours in March (labor usage in March is 2,355 hours as compared to the base employment level of 2,240 hours). This suggests that perhaps Greenberg should consider increasing the base employment level. Storage space is not an issue (at least during the

SCREENSHOT 3-11A **Excel Layout and Solver Entries for Greenberg Motors**

All entries in column R are computed using the SUMPRODUCT function.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
1	**Greenberg Motors**																			
2																				
3		P_{A1}	I_{A1}	P_{A2}	I_{A2}	P_{A3}	I_{A3}	P_{A4}	I_{A4}	P_{B1}	I_{B1}	P_{B2}	I_{B2}	P_{B3}	I_{B3}	P_{B4}	I_{B4}			
4		GM3A Jan prod	GM3A Jan inv	GM3A Feb prod	GM3A Feb inv	GM3A Mar prod	GM3A Mar inv	GM3A Apr prod	GM3A Apr inv	GM3B Jan prod	GM3B Jan inv	GM3B Feb prod	GM3B Feb inv	GM3B Mar prod	GM3B Mar inv	GM3B Apr prod	GM3B Apr inv			
5	Number of Units	1,276.92	476.92	1,138.46	915.38	842.31	757.69	792.31	450.00	1,000.00	0.00	1,200.00	0.00	1,400.00	0.00	1,700.00	300.00			
6	Cost	$10.00	$0.18	$10.00	$0.18	$11.00	$0.18	$11.00	$0.18	$6.00	$0.13	$6.00	$0.13	$6.60	$0.13	$6.60	$0.13	$76,301.62		
7	**Constraints:**																			
8	GM3A Jan balance	1	-1															800.00	=	800
9	GM3B Jan balance									1	-1							1000.00	=	1000
10	GM3A Feb balance		1	1	-1													700.00	=	700
11	GM3B Feb balance										1	1	-1					1200.00	=	1200
12	GM3A Mar balance				1	1	-1											1000.00	=	1000
13	GM3B Mar balance												1	1	-1			1400.00	=	1400
14	GM3A Apr balance						1	1	-1									1100.00	=	1100
15	GM3B Apr balance														1	1	-1	1400.00	=	1400
16	GM3A Apr Inventory								1									450.00	=	450
17	GM3B Apr Inventory																1	300.00	=	300
18	Jan storage cap		1								1							476.92	<=	3300
19	Feb storage cap				1								1					915.38	<=	3300
20	Mar storage cap						1								1			757.69	<=	3300
21	Apr storage cap								1								1	750.00	<=	3300
22	Jan labor max	1.3								0.9								2560.00	<=	2560
23	Feb labor max			1.3								0.9						2560.00	<=	2560
24	Mar labor max					1.3								0.9				2355.00	<=	2560
25	Apr labor max							1.3								0.9		2560.00	<=	2560
26	Jan labor min	1.3								0.9								2560.00	>=	2240
27	Feb labor min			1.3								0.9						2560.00	>=	2240
28	Mar labor min					1.3								0.9				2355.00	>=	2240
29	Apr labor min							1.3								0.9		2560.00	>=	2240
30																		**LHS**	**Sign**	**RHS**

Solver Parameters

Set Objective: $R$6

To: ○ Max ◉ Min ○ Value Of:

By Changing Variable Cells:
$B$5:$Q$5

Subject to the Constraints:
$R$18:$R$25 <= $T$18:$T$25
$R$26:$R$29 >= $T$26:$T$29
$R$8:$R$17 = $T$8:$T$17

Solver includes three entries: One for each type of constraints ≤, ≥, and =.

TABLE 3.12 **Solution to Greenberg Motors Problem**

PRODUCTION SCHEDULE	JANUARY	FEBRUARY	MARCH	APRIL
Units of GM3A produced	1,276.92	1,138.46	842.31	792.31
Units of GM3B produced	1,000.00	1,200.00	1,400.00	1,700.00
Inventory of GM3A carried	476.92	915.38	757.69	450.00
Inventory of GM3B carried	0.00	0.00	0.00	300.00
Labor hours required	2,560.00	2,560.00	2,355.00	2,560.00

File: 3-11.xls, sheet: 3-11B

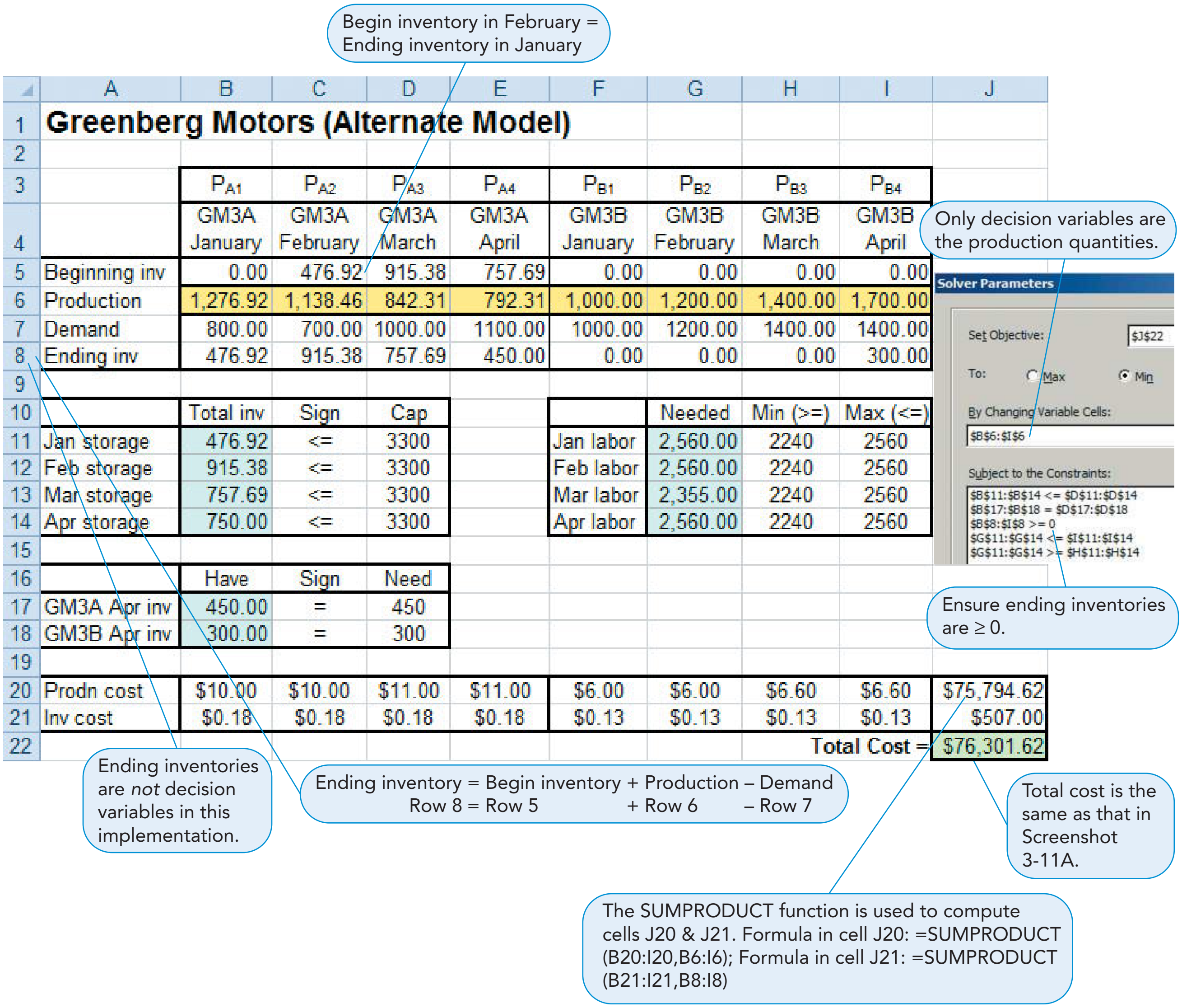
Ending inventory is not set as a decision variable in this alternate Excel layout for the Greenberg Motors problem.

four months in consideration here), with less than one-third of the available space being used each month. The solution does include several fractional values, which may need to be rounded off appropriately before implementation. Alternatively, in such problem scenarios, it may be possible to view fractional production values as work-in-process (WIP) inventories.

For many multiperiod problems, it may often be convenient to group all the variables for a given month in the same column. Screenshot 3-11B shows the Excel layout of an alternate model for the Greenberg Motors problem.

Note that in this alternate model, the only decision variables are the production variables (P_{A1} to P_{A4}, P_{B1} to P_{B4}). The inventory variables are no longer explicitly stated as decision variables,

SCREENSHOT 3-11B **Excel Layout and Solver Entries for Greenberg Motors—Alternate Model**



	A	B	C	D	E	F	G	H	I	J
1	**Greenberg Motors (Alternate Model)**									
2										
3		P_{A1}	P_{A2}	P_{A3}	P_{A4}	P_{B1}	P_{B2}	P_{B3}	P_{B4}	
4		GM3A January	GM3A February	GM3A March	GM3A April	GM3B January	GM3B February	GM3B March	GM3B April	
5	Beginning inv	0.00	476.92	915.38	757.69	0.00	0.00	0.00	0.00	
6	Production	1,276.92	1,138.46	842.31	792.31	1,000.00	1,200.00	1,400.00	1,700.00	
7	Demand	800.00	700.00	1000.00	1100.00	1000.00	1200.00	1400.00	1400.00	
8	Ending inv	476.92	915.38	757.69	450.00	0.00	0.00	0.00	300.00	
9										
10		Total inv	Sign	Cap			Needed	Min (>=)	Max (<=)	
11	Jan storage	476.92	<=	3300		Jan labor	2,560.00	2240	2560	
12	Feb storage	915.38	<=	3300		Feb labor	2,560.00	2240	2560	
13	Mar storage	757.69	<=	3300		Mar labor	2,355.00	2240	2560	
14	Apr storage	750.00	<=	3300		Apr labor	2,560.00	2240	2560	
15										
16		Have	Sign	Need						
17	GM3A Apr inv	450.00	=	450						
18	GM3B Apr inv	300.00	=	300						
19										
20	Prodn cost	$10.00	$10.00	$11.00	$11.00	$6.00	$6.00	$6.60	$6.60	$75,794.62
21	Inv cost	$0.18	$0.18	$0.18	$0.18	$0.13	$0.13	$0.13	$0.13	$507.00
22									Total Cost =	$76,301.62

and they are not specified as changing variable cells in Solver. Rather, they are calculated as simple by-products of the other parameters in the problem. Using the standard inventory constraints, the ending inventory each month is calculated as follows:

$$\begin{pmatrix}\text{inventory}\\ \text{at the}\\ \text{end of}\\ \text{last month}\end{pmatrix} + \begin{pmatrix}\text{current}\\ \text{month's}\\ \text{production}\end{pmatrix} - \begin{pmatrix}\text{current}\\ \text{month's}\\ \text{sales}\end{pmatrix} = \begin{pmatrix}\text{inventory}\\ \text{at the}\\ \text{end of}\\ \text{this month}\end{pmatrix}$$

$$\text{Row 5} \quad + \quad \text{Row 6} \quad - \quad \text{Row 7} \quad = \quad \text{Row 8}$$

Because they are no longer decision variables, however, we need to add constraints to ensure that the ending inventories for all products have nonnegative values in each month. Depending on individual preferences and expertise, we can design other layouts for setting up and solving this problem using Excel.

The Greenberg Motors example illustrates a relatively simple production planning problem in that only two products were considered for a four-month planning horizon. The LP model discussed here can, however, be applied successfully to problems with dozens of products, hundreds of constraints, and longer planning horizons.

Sinking Fund Problem

Another excellent example of a multiperiod problem is the sinking fund problem. In this case, an investor or a firm seeks to establish an investment portfolio, using the least possible initial investment, that will generate specific amounts of capital at specific time periods in the future.

Consider the example of Larry Fredendall, who is trying to plan for his daughter Susan's college expenses. Based on current projections (it is now the start of year 1), Larry anticipates that his financial needs at the start of each of the following years is as shown in Table 3.13.

Larry has several investment choices to choose from at the present time, as listed in Table 3.14. Each choice has a fixed known return on investment and a specified maturity date. Assume that each choice is available for investment at the start of every year and also assume that returns are tax free if used for education. Because choices C and D are relatively risky choices, Larry wants no more than 20% of his total investment in those two choices at any point in time.

Larry wants to establish a sinking fund to meet his requirements. Note that at the start of year 1, the entire initial investment is available for investing in the choices. However, in subsequent years, only the amount maturing from a prior investment is available for investment.

FORMULATING THE PROBLEM Let us first define the decision variables. Note that in defining these variables, we need to consider only those investments that will mature by the end of year 5, at the latest, because there is no requirement after 6 years:

$$
\begin{aligned}
A_1 &= \text{\$ amount invested in choice A at the start of year 1} \\
B_1 &= \text{\$ amount invested in choice B at the start of year 1} \\
C_1 &= \text{\$ amount invested in choice C at the start of year 1} \\
D_1 &= \text{\$ amount invested in choice D at the start of year 1} \\
A_2 &= \text{\$ amount invested in choice A at the start of year 2} \\
B_2 &= \text{\$ amount invested in choice B at the start of year 2} \\
C_2 &= \text{\$ amount invested in choice C at the start of year 2} \\
D_2 &= \text{\$ amount invested in choice D at the start of year 2} \\
A_3 &= \text{\$ amount invested in choice A at the start of year 3} \\
B_3 &= \text{\$ amount invested in choice B at the start of year 3} \\
C_3 &= \text{\$ amount invested in choice C at the start of year 3} \\
A_4 &= \text{\$ amount invested in choice A at the start of year 4} \\
B_4 &= \text{\$ amount invested in choice B at the start of year 4} \\
A_5 &= \text{\$ amount invested in choice A at the start of year 5}
\end{aligned}
$$

The objective is to minimize the initial investment and can be expressed as

$$\text{Minimize } A_1 + B_1 + C_1 + D_1$$

TABLE 3.13
Financial Needs for Larry Fredendall

YEAR	$ NEEDED
3	$20,000
4	$22,000
5	$24,000
6	$26,000

TABLE 3.14
Investment Choices for Larry Fredendall

CHOICE	ROI	MATURITY
A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

As in the multiperiod production scheduling problem, we need to write balance constraints for each period (year). These constraints recognize the relationship between the investment decisions made in any given year and the investment decisions made in all prior years. Specifically, we need to ensure that the amount used for investment at the start of a given year is restricted to the amount maturing at the end of the previous year *less* any payments made for Susan's education that year. This relationship can be modeled as

$$\begin{pmatrix}\text{amount}\\ \text{invested at}\\ \text{start of}\\ \text{year } t\end{pmatrix} + \begin{pmatrix}\text{amount}\\ \text{paid for}\\ \text{education at}\\ \text{start of year } t\end{pmatrix} = \begin{pmatrix}\text{amound}\\ \text{maturing}\\ \text{at end}\\ \text{of year } (t-1)\end{pmatrix}$$

This equation is analogous to the inventory equations in the production scheduling problem.

At the start of year 2, the total amount maturing is $1.05A_1$ (investment in choice A in year 1 plus 5% interest). The constraint at the start of year 2 can therefore be written as

$$A_2 + B_2 + C_2 + D_2 = 1.05A_1 \qquad \text{(year 2 cash flow)}$$

These are the cash flow constraints.

Constraints at the start of years 3 through 6 are as follows and also include the amounts payable for Susan's education each year:

$$\begin{aligned}
A_3 + B_3 + C_3 + 20{,}000 &= 1.13B_1 + 1.05A_2 && \text{(year 3 cash flow)}\\
A_4 + B_4 + 22{,}000 &= 1.28C_1 + 1.13B_2 + 1.05A_3 && \text{(year 4 cash flow)}\\
A_5 + 24{,}000 &= 1.4D_1 + 1.28C_2 + 1.13B_3 + 1.05A_4 && \text{(year 5 cash flow)}\\
26{,}000 &= 1.4D_2 + 1.28C_3 + 1.13B_4 + 1.05A_5 && \text{(year 6 cash flow)}
\end{aligned}$$

These five constraints address the cash flow issues. However, they do not account for Larry's risk preference with regard to investments in choices C and D in any given year. To satisfy these requirements, we need to ensure that total investment in choices C and D in any year is no more than 20% of the total investment in *all* choices that year. In keeping track of these investments, it is important to also account for investments in *prior* years that may have still not matured. At the start of year 1, this constraint can be written as

$$C_1 + D_1 \le 0.2(A_1 + B_1 + C_1 + D_1) \qquad \text{(year 1 risk)}$$

These are the risk preference constraints.

In writing this constraint at the start of year 2, we must take into account the fact that investments B_1, C_1, and D_1 have still not matured. Therefore,

$$C_1 + D_1 + C_2 + D_2 \le 0.2(B_1 + C_1 + D_1 + A_2 + B_2 + C_2 + D_2) \qquad \text{(year 2 risk)}$$

Constraints at the start of years 3 through 5 are as follows. Note that there is no constraint necessary at the start of year 6 because there are no investments that year:

$$\begin{aligned}
C_1 + D_1 + C_2 + D_2 + C_3 &\le 0.2(C_1 + D_1 + B_2 + C_2 + D_2 + A_3 + B_3 + C_3) && \text{(year 3 risk)}\\
D_1 + C_2 + D_2 + C_3 &\le 0.2(D_1 + C_2 + D_2 + B_3 + C_3 + A_4 + B_4) && \text{(year 4 risk)}\\
D_2 + C_3 &\le 0.2(D_2 + C_3 + B_4 + A_5) && \text{(year 5 risk)}
\end{aligned}$$

Finally, we have the nonnegativity constraints:

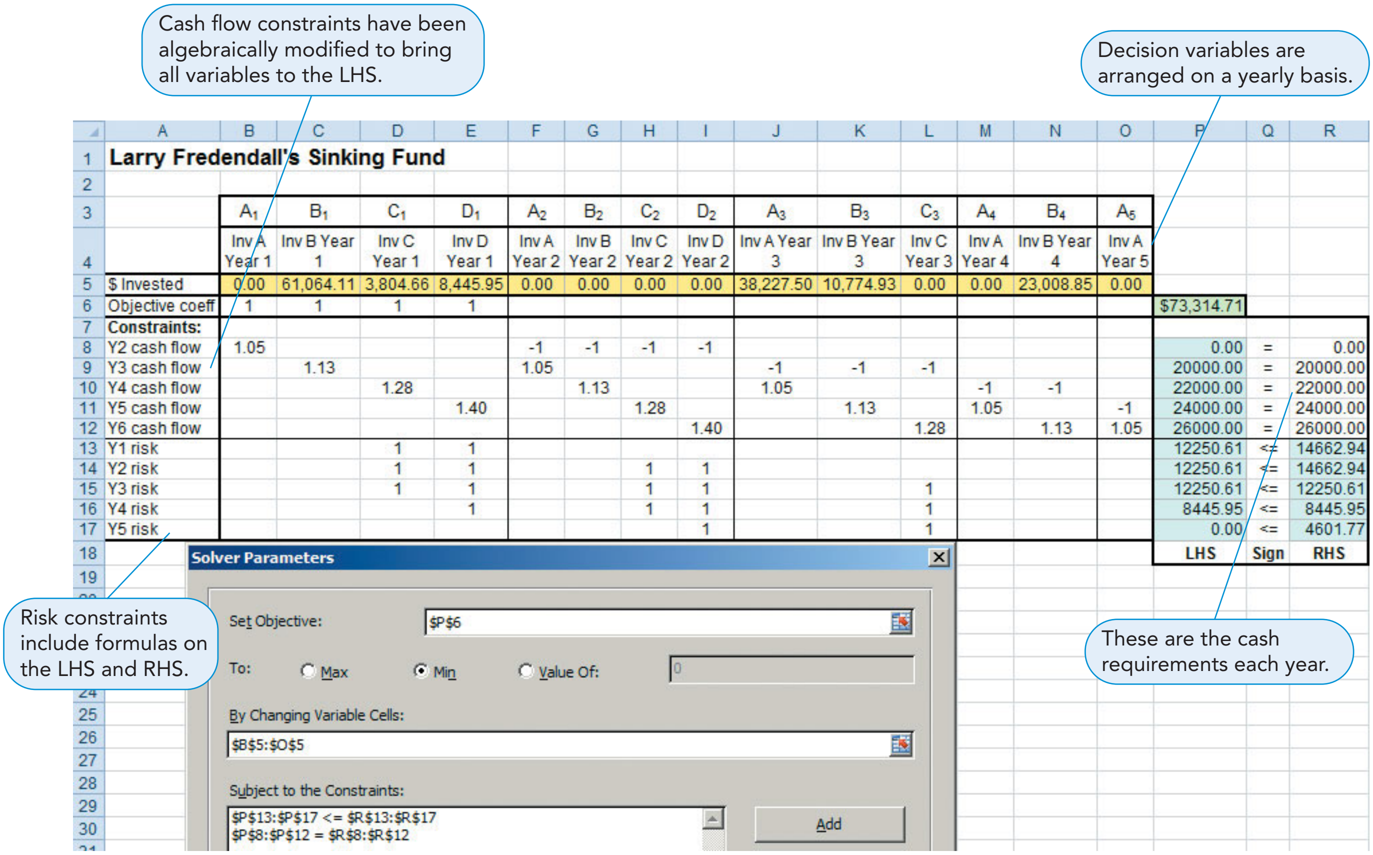
$$\text{All variables} \ge 0$$

File: 3-12.xls

SOLVING THE PROBLEM AND INTERPRETING THE RESULTS Screenshot 3-12 shows the Excel layout and Solver entries for this model. As with the production scheduling problem, there are several alternate ways in which the Excel layout could be structured, depending on the preference and expertise of the analyst. In our implementation of this model, we have algebraically modified the cash flow constraints for each year so that all variables are on the LHS and the education cash outflows are on the RHS. We have, however, implemented the risk constraints as written in the formulation above. The modified cash flow constraints are as follows:

$$\begin{aligned}
1.05A_1 - A_2 - B_2 - C_2 - D_2 &= 0 && \text{(year 2 cash flow)}\\
1.13B_1 + 1.05A_2 - A_3 - B_3 - C_3 &= 20{,}000 && \text{(year 3 cash flow)}\\
1.28C_1 + 1.13B_2 + 1.05A_3 - A_4 - B_4 &= 22{,}000 && \text{(year 4 cash flow)}\\
1.4D_1 + 1.28C_2 + 1.13B_3 + 1.05A_4 - A_5 &= 24{,}000 && \text{(year 5 cash flow)}\\
1.4D_2 + 1.28C_3 + 1.13B_4 + 1.05A_5 &= 26{,}000 && \text{(year 6 cash flow)}
\end{aligned}$$

SCREENSHOT 3-12 Excel Layout and Solver Entries for Larry Fredendall's Sinking Fund



The optimal solution requires Larry to invest a total of $73,314.71 at the start of year 1, putting $61,064.11 in choice B, $3,804.66 in choice C, and $8,445.95 in choice D. There is no money maturing for investment at the start of year 2. At the start of year 3, using the maturing amounts, Larry should pay off $20,000 for Susan's education, invest $38,227.50 in choice A, and invest $10,774.93 in choice B. At the start of year 4, Larry should use the maturing amounts to pay off $22,000 for Susan's education and invest $23,008.85 in choice B. The investments in place at that time will generate $24,000 at the start of year 5 and $26,000 at the start of year 6, meeting Larry's requirements in those years.

Summary

This chapter continues the discussion of LP models. To show ways of formulating and solving problems from a variety of disciplines, we examine applications from manufacturing, marketing, finance, employee scheduling, transportation, ingredient blending, and multiperiod planning. We also consider an example with a special type of objective function. For each example, we illustrate how to set up and solve all these models by using Excel's Solver add-in.

Solved Problem

Solved Problem 3-1

The Loughry Group has opened a new mall in Gainesville, Florida. Mark Loughry, the general manager of the mall, is trying to ensure that enough support staff are available to clean the mall before it opens each day. The mall operates seven days a week, and the cleaning staff work between 12:30 A.M. and 8:30 A.M. each night. Based on projected mall traffic data for the upcoming week, Mark estimates that the number of cleaning staff required each day will be as shown in Table 3.15.

TABLE 3.15 Cleaning Staff Requirement Data for Loughry Group's Mall

DAY OF WEEK	NUMBER OF STAFF REQUIRED
Monday	22
Tuesday	13
Wednesday	15
Thursday	20
Friday	18
Saturday	23
Sunday	27

TABLE 3.16 Schedule and Cost Data for Loughry Group's Mall

WORK SCHEDULE	WAGES PER WEEK
1. Saturday and Sunday off	\$350
2. Saturday and Tuesday off	\$375
3. Tuesday and Wednesday off	\$400
4. Monday and Thursday off	\$425
5. Tuesday and Friday off	\$425
6. Thursday and Friday off	\$400
7. Sunday and Thursday off	\$375
8. Sunday and Wednesday off	\$375

Mark can use the work schedules shown in Table 3.16 for the cleaning staff. The wages for each schedule are also shown in Table 3.16. In order to be perceived as being a fair employer, Mark wants to ensure that at least 75% of the workers have two consecutive days off and that at least 50% of the workers have at least one weekend day off. How should Mark schedule his cleaning staff in order to meet the mall's requirements?

Solution

FORMULATING AND SOLVING THE PROBLEM As noted in the previous labor staffing problem, the decision variables typically determine how many employees need to start their work at the different starting times permitted. In Mark's case, because there are eight possible work schedules, we have eight decision variables in the problem. Let

S_1 = number of employees who need to follow schedule 1 (Saturday and Sunday off)
S_2 = number of employees who need to follow schedule 2 (Saturday and Tuesday off)
S_3 = number of employees who need to follow schedule 3 (Tuesday and Wednesday off)
S_4 = number of employees who need to follow schedule 4 (Monday and Thursday off)
S_5 = number of employees who need to follow schedule 5 (Tuesday and Friday off)
S_6 = number of employees who need to follow schedule 6 (Thursday and Friday off)
S_7 = number of employees who need to follow schedule 7 (Sunday and Thursday off)
S_8 = number of employees who need to follow schedule 8 (Sunday and Wednesday off)

This is the objective function:

$$\text{Minimize total weekly wages} = \$350\,S_1 + \$375\,S_2 + \$400\,S_3 + \$425\,S_4 + \$425\,S_5 + \$400\,S_6 + \$375\,S_7 + \$375\,S_8$$

Subject to the constraints

$$S_1 + S_2 + S_3 + S_5 + S_6 + S_7 + S_8 \geq 22 \quad \text{(Monday requirement)}$$
$$S_1 + S_2 + S_4 + S_6 + S_7 + S_8 \geq 13 \quad \text{(Tuesday requirement)}$$
$$S_1 + S_2 + S_4 + S_5 + S_6 + S_7 \geq 15 \quad \text{(Wednesday requirement)}$$

$$
\begin{aligned}
S_1 + S_2 + S_3 + S_5 + S_8 &\geq 20 &&\text{(Thursday requirement)}\\
S_1 + S_2 + S_3 + S_4 + S_7 + S_8 &\geq 18 &&\text{(Friday requirement)}\\
S_3 + S_4 + S_5 + S_6 + S_7 + S_8 &\geq 23 &&\text{(Saturday requirement)}\\
S_2 + S_3 + S_4 + S_5 + S_6 &\geq 27 &&\text{(Sunday requirement)}
\end{aligned}
$$

At least 75% of workers must have two consecutive days off each week, and at least 50% of workers must have at least one weekend day off each week. These constraints may be written, respectively, as

$$
\begin{aligned}
S_1 + S_3 + S_6 &\geq 0.75\,(S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8)\\
S_1 + S_2 + S_7 + S_8 &\geq 0.5\,(S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8)
\end{aligned}
$$

Finally,

$$S_1, S_2, S_3, S_5, S_6, S_7, S_8 \geq 0$$

WWW

File: 3-13.xls

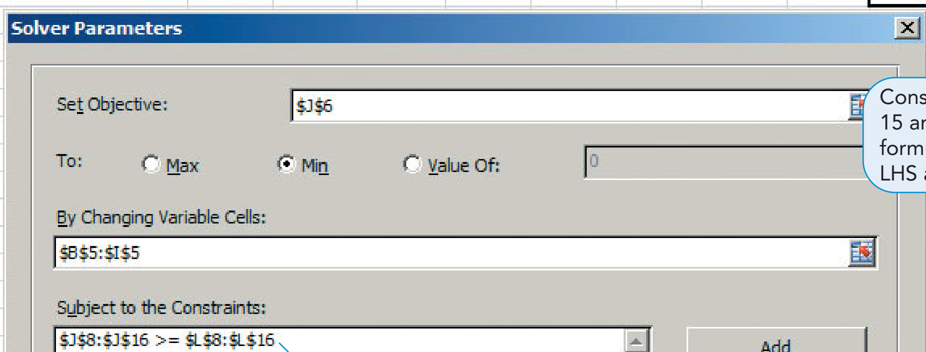
The Excel layout and Solver entries for this model are shown in Screenshot 3-13. For the last two constraints, the Excel layout includes formulas for both the LHS (cells J15:J16) and RHS (cells L15:L16) entries. While the formulas in cells J15:J16 use the usual SUMPRODUCT function, the formulas in cells L15:L16 are

Cell L15: =0.75*SUM($B$5:$I$5)

Cell L16: =0.50*SUM($B$5:$I$5)

SCREENSHOT 3-13 Excel Layout and Solver Entries for Loughry Group Mall

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Loughry Group Mall**											
2												
3		S1	S2	S3	S4	S5	S6	S7	S8			
4		Sat & Sun off	Sat & Tue off	Tue & Wed off	Mon & Thu off	Tue & Fri off	Thu & Fri off	Sun & Thu off	Sun & Wed off			
5	Number of staff	10.00	7.00	20.00	0.00	0.00	0.00	0.00	3.00			
6	Wages	$350	$375	$400	$425	$425	$400	$375	$375	$15,250.00		
7	**Constraints:**											
8	Monday needs	1	1	1		1	1	1	1	40.00	>=	22
9	Tuesday needs	1			1		1	1	1	13.00	>=	13
10	Wednesday needs	1	1		1	1	1	1		17.00	>=	15
11	Thursday needs	1	1	1		1			1	40.00	>=	20
12	Friday needs	1	1	1	1			1	1	40.00	>=	18
13	Saturday needs			1	1	1	1	1	1	23.00	>=	23
14	Sunday needs		1	1	1	1	1			27.00	>=	27
15	75% consecutive	1		1			1			30.00	>=	30
16	50% weekend day	1	1					1	1	20.00	>=	20
17										**LHS**	**Sign**	**RHS**



Constraints in rows 15 and 16 include formulas on both LHS and RHS.

All nine ≥ constraints entered as a single entry in Solver.

INTERPRETING THE RESULTS Screenshot 3-13 reveals that the optimal solution is to employ 10 people on schedule 1; 7 people on schedule 2; 20 people on schedule 3; and 3 people on schedule 8, for a total cost of $15,250 per week. As in the solution for the Hong Kong Bank problem (see Screenshot 3-6 on page 83), it turns out that this problem has alternate optimal solutions, too. For example, we can satisfy the staff requirements at the same cost by employing 10 people on schedule 1; 7 people on schedule 2; 3 people on schedule 3; 17 people on schedule 6; and 3 people on schedule 7.

The solution indicates that while exactly meeting staffing needs for Tuesday, Saturday, and Sunday, Mark is left with way more than he needs on Monday, Thursday, and Friday. He should perhaps consider using part-time help to alleviate this mismatch in his staffing needs.

Problems

3-1 A small backpack manufacturer carries four different models of backpacks, made of canvas, plastic, nylon, and leather. The bookstore, which will exclusively sell the backpacks, expects to be able to sell between 15 and 40 of each model. The store has agreed to pay $35.50 for each canvas backpack, $39.50 for each plastic backpack, $42.50 for each nylon backpack, and $69.50 for each leather backpack that can be delivered by the end of the following week.

One worker can work on either canvas or plastic, can complete a backpack in 1.5 hours, and will charge $7.00 per hour to do the work. This worker can work a maximum of 90 hours during the next week. Another worker can sew backpacks made of nylon fabric. He can complete a bag in 1.7 hours, will charge $8.00 per hour to work, and can work 42.5 hours in the next week. A third worker has the ability to sew leather. He each can complete a book bag in 1.9 hours, will charge $9.00 per hour to work, and can work 80 hours during the next week. The following table provides additional information about each backpack. What is the best combination of backpacks to provide the store to maximize the profit?

BACKPACK MODEL	MATERIAL REQUIRED (SQUARE YARDS)	MATERIAL AVAILABLE (SQUARE YARDS)	COST/ SQUARE YARD
Canvas	2.25	200	$4.50
Plastic	2.40	350	$4.25
Nylon	2.10	700	$7.65
Leather	2.60	550	$9.45

3-2 A contestant on the hit reality television show Top Bartender was asked to mix a variety of drinks, each consisting of 4 fluid ounces. No other ingredients were permitted. She was given the following quantities of liquor:

LIQUOR	QUANTITY
Bourbon	128 ounces
Brandy	128 ounces
Vodka	128 ounces
Dry Vermouth	32 ounces
Sweet Vermouth	32 ounces

The contestant is considering making the following four drinks:

- The New Yorker: 25% each of bourbon, brandy, vodka, and sweet vermouth
- The Garaboldi: 25% each of brandy and dry vermouth; 50% sweet vermouth
- The Kentuckian: 100% bourbon
- The Russian: 75% vodka and 25% dry vermouth

The contestant's objective is to make the largest number of drinks with the available liquor. What is the combination of drinks to meet her objective?

3-3 A manufacturer of travel pillows must determine the production plan for the next production cycle. He wishes to make at least 300 of each of the three models that his firm offers and no more than 1,200 of any one model. The specifics for each model are shown in the following table. How many pillows of each type should be manufactured in order to maximize total profit?

PILLOW MODEL	SELLING PRICE	CUTTING	SEWING	FINISHING	PACKING
Junior travel pillow	$5.75	0.10	0.05	0.18	0.20
Travel pillow	$6.95	0.15	0.12	0.24	0.20
Deluxe travel pillow	$7.50	0.20	0.18	0.20	0.20
Available hours		450	550	600	450
Cost per hour		$7.00	$9.00	$8.50	$7.25

Table for Problem 3-5

CABINET STYLE	CARPENTRY	PAINTING	FINISHING	PROFIT
Italian	3.00	1.50	0.75	\$72
French	2.25	1.00	0.75	\$65
Caribbean	2.50	1.25	0.85	\$78
Available hours	1,360	700	430	

3-4 Students who are trying to raise funds have an agreement with a local pizza chain. The chain has agreed to sell them pizzas at a discount, which the students can then resell to families in the local community for a profit. It is expected that of the 500 families in the community, at most 70% will buy pizza. Based on a survey of their personal preferences, the students believe that they should order no more than 120 cheese pizzas, no more than 150 pepperoni pizzas, and no more than 100 vegetarian pizzas. They also want to make sure that at least 20% of the total pizzas are cheese and at least 50% of the pizzas are pepperoni. They make a profit of \$1.45, \$1.75, and \$1.98, respectively, for each cheese, pepperoni, and vegetarian pizza they resell. How many pizzas of each type should they buy?

3-5 A furniture maker sells three different styles of cabinets, including an Italian model, a French Country model, and a Caribbean model. Each cabinet produced must go through three departments: carpentry, painting, and finishing. The table at the top of this page contains all relevant information concerning production times (hours per cabinet), production capacities for each operation per day, and profit (\$ per unit). The owner has an obligation to deliver a minimum of 60 cabinets in each style to a furniture distributor. He would like to determine the product mix that maximizes his daily profit. Formulate the problem as an LP model and solve using Excel.

3-6 An electronics manufacturer has an option to produce six styles of cell phones. Each of these devices requires time, in minutes, on three types of electronic testing equipment, as shown in the table at the bottom of this page. The first two test devices are each available for 120 hours per week. Test device 3 requires more preventive maintenance and may be used only for 100 hours each week. The market for all six cell phones is vast, so the manufacturer believes that it can sell as many cell phones as it can manufacture. The table also summarizes the revenues and material costs for each type of phone.

In addition, variable labor costs are \$15 per hour for test device 1, \$12 per hour for test device 2, and \$18 per hour for test device 3. Determine the product mix that would maximize profits. Formulate the problem as an LP model and solve it by using Excel.

3-7 A company produces three different types of wrenches: W111, W222, and W333. It has a firm order for 2,000 W111 wrenches, 3,750 W222 wrenches, and 1,700 W333 wrenches. Between now and the order delivery date, the company has only 16,500 fabrication hours and 1,600 inspection hours. The time that each wrench requires in each department is shown in the table at the bottom of this page. Also shown are the costs to manufacture the wrenches in-house and the costs to outsource them. For labeling considerations, the company wants to manufacture in-house at least 60% of each type of wrench that will be shipped. How many wrenches of each type should be made in-house and how many should be

Table for Problem 3-6

	SMARTPHONE	BLUEBERRY	MOPHONE	BOLDPHONE	LUXPHONE4G	TAP3G
Test device 1	7	3	12	6	18	17
Test device 2	2	5	3	2	15	17
Test device 3	5	1	3	2	9	2
Revenue per unit	\$200	\$120	\$180	\$200	\$430	\$260
Material cost per unit	\$35	\$25	\$40	\$45	\$170	\$60

Table for Problem 3-7

WRENCH	FABRICATION HOURS	INSPECTION HOURS	IN-HOUSE COST	OUTSOURCE COST
W111	2.50	0.25	\$17.00	\$20.40
W222	3.40	0.30	\$19.00	\$21.85
W333	3.80	0.45	\$23.00	\$25.76

outsourced? What will be the total cost to satisfy the order?

3-8 A gear manufacturer is planning next week's production run for four types of gears. If necessary, it is possible to outsource any type of gear from another gear company located nearby. The following table and the table at the bottom of this page show next week's demand, revenue per unit, outsource cost per unit, time (in hours) required per unit in each production process, and the availability and costs of these processes. The nearby company can supply a maximum of 300 units of each type of gear next week. What should be the production and/or outsource plan for the next week to maximize profit?

GEAR TYPE	GEAR A	GEAR B	GEAR C	GEAR D
Demand	400	500	450	600
Revenue	\$12.50	\$15.60	\$17.40	\$19.30
Outsource	\$7.10	\$8.10	\$8.40	\$9.00

3-9 A political polling organization is to conduct a poll of likely voters prior to an upcoming election. Each voter is to be interviewed in person. It is known that the costs of interviewing different types of voters vary due to the differences in proportion within the population. The costs to interview males, for example, are \$10 per Democrat, \$9 per Republican, and \$13.50 per Independent voter. The costs to interview females are \$12, \$11 and \$13.50 for Democrat, Republican, and Independent voters, respectively. The polling service has been given certain criteria to which it must adhere:

- There must be at least 4,500 total interviews.
- At least 1,000 independent voters must be polled.
- At least 2,000 males must be polled.
- At least 1,750 females must be polled.
- No more than 40% of those polled may be Democrats.
- No more than 40% of those polled may be Republicans.
- No more than one-quarter of those polled may be Republican males.
- Each of the six categories of voters must be represented in the poll by at least 10% of the total interviews.

Determine the least expensive sampling plan and the total cost to carry out the plan.

3-10 The advertising director a large retail store in Columbus, Ohio, is considering three advertising media possibilities: (1) ads in the Sunday *Columbus Dispatch* newspaper, (2) ads in a local trade magazine that is distributed free to all houses in the city and northwest suburbs, and (3) ads on Columbus' WCC-TV station. She wishes to obtain a new-customer exposure level of at least 50% within the city and 60% in the northwest suburbs. Each TV ad has a new-customer exposure level of 5% in the city and 3% in the northwest suburbs. The *Dispatch* ads have corresponding exposure levels per ad of 3.5% and 3%, respectively, while the trade magazine has exposure levels per ad of 0.5% and 1%, respectively. The relevant costs are \$1,000 per *Dispatch* ad, \$300 per trade magazine ad, and \$2,000 per TV ad. The advertising policy is that no single media type should consume more than 45% of the total amount spent. Find the advertising strategy that will meet the store's objective at minimum cost.

3-11 A grocery chain wants to promote the sale of a new flavor of ice cream by issuing up to 15,000 coupons by mail to preferred customers. The budget for this promotion has been limited to \$12,000. The following table shows the expected increased sales per coupon and the probability of coupon usage for the various coupon amounts under consideration.

COUPON AMOUNT	INCREASED SALES PER COUPON (CARTONS)	PROBABILITY COUPON WILL BE USED
\$1.00	1.50	0.80
\$0.85	1.40	0.75
\$0.70	1.25	0.60
\$0.55	1.00	0.50
\$0.40	0.90	0.42

For example, every \$1-off coupon issued will stimulate sales of 1.5 additional cartons. However, since the probability that a \$1-off coupon will actually be used is only 0.80, the expected increased sales per coupon issued is 1.2 $(= 0.8 \times 1.5)$ cartons.

The selling price per carton of ice cream is \$3.50 before the coupon value is applied. The chain wants at least 20% of the coupons issued to be of the \$1-off variety and at least 10% of the coupons issued to be

Table for Problem 3-8

PROCESS	GEAR A	GEAR B	GEAR C	GEAR D	HOURS AVAILABLE	COST PER HOUR
Forming	0.30	0.36	0.38	0.45	500	\$9.00
Hardening	0.20	0.30	0.24	0.33	300	\$8.00
Deburring	0.30	0.30	0.35	0.25	310	\$7.50

of each of the other four varieties. What is the optimal combination of coupons to be issued, and what is the expected net increased revenue from this promotion?

3-12 A political candidate is planning his media budget for an upcoming election. He has $90,500 to spend. His political consultants have provided him with the following estimates of additional votes as a result of the advertising effort:

- For every small sign placed by the roadside, he will garner 10 additional votes.
- For every large sign placed by the roadside, he will garner 30 additional votes.
- For every thousand bumper stickers placed on cars, he will garner 10 additional votes.
- For every hundred personal mailings to registered voters, he will garner 40 additional votes, and
- For every radio ad heard daily in the last month before the election, he will garner 485 additional votes.

The costs for each of these advertising devices, along with the practical minimum and maximum that should be planned for each, are shown on the following table. How should the candidate plan to spend his campaign money?

ADVERTISING MEDIUM	COST	MINIMUM	MAXIMUM
Bumper stickers (thousands)	$30	40	100
Personal mailings (hundreds)	$81	500	800
Radio ads (per day)	$1,000	3	12
Small road side signs	$25	100	500
Large road side signs	$60	50	300

3-13 A brokerage firm has been tasked with investing $500,000 for a new client. The client has asked that the broker select promising stocks and bonds for investment, subject to the following guidelines:

- At least 20% in municipal bonds
- At least 10% each in real estate stock and pharmaceutical stock
- At least 40% in a combination of energy and domestic automobile stocks, with each accounting for at least 15%
- No more than 50% of the total amount invested in energy and automobile stocks in a combination of real estate and pharmaceutical company stock

Subject to these constraints, the client's goal is to maximize projected return on investments. The broker has prepared a list of high-quality stocks and bonds and their corresponding rates of return, as shown in the following table.

INVESTMENT	ANNUAL RATE OF RETURN
City of Miami (municipal) bonds	5.3%
American Smart Car	8.8%
GreenEarth Energy	4.9%
Rosslyn Pharmaceuticals	8.4%
RealCo (real estate)	10.4%

Formulate this portfolio selection problem by using LP and solve it by using Excel.

3-14 An investor wishes to invest some or all of his $12.5 million in a diversified portfolio through a commercial lender. The types of investments, the expected interest per year, and the maximum allowed percentage investment he will consider are shown on the following table. He wants at least 35% of his investments to be in nonmortgage instruments and no more than 60% to be in high-yield (and high-risk) instruments (i.e., expected interest >8%). How should his investment be diversified to make the most interest income?

INVESTMENT	EXPECTED INTEREST	MAXIMUM ALLOWED
Low-income mortgage loans	7.00%	20%
Conventional mortgage loans	6.25%	40%
Government sponsored mortgage loans	8.25%	25%
Bond investments	5.75%	12%
Stock investments	8.75%	15%
Futures trading	9.50%	10%

3-15 A finance major has inherited $200,000 and wants to invest it in a diversified portfolio. Some of the investments she is considering are somewhat risky. These include international mutual funds, which should earn 12.25% over the next year, and U.S. stocks, which should earn 11.5% over the next year. She has therefore decided that she will put no more than 30% of her money in either of these investments and no more than a total of 50% in both investments.

She also wants to keep some of her investment in what is considered a liquid state, so that she can divest quickly if she so chooses. She believes school bonds, which return 5% interest, short-term certificates of deposit, which return 6.25% interest, and tax-free municipal bonds, which return 8.75%, to be reasonably liquid. She will keep no more than 40% of her money in these investments and no more than 15% in any one of these investments. She believes that T-bills are also considered liquid and less risky and that they will return 7.5%. However, she has decided to invest no more than 25% of her investment in T-bills.

She wishes to have experience investing in different types of instruments, so she will invest at least 10% of her money in each of the six types of investment choices. What is the optimal investment strategy for her to follow?

3-16 A couple has agreed to attend a "casino night" as part of a fundraiser for the local hospital, but they believe that gambling is generally a losing proposition. For the sake of the charity, they have decided to attend and to allocate $300 for the games. There are to be four games, each involving standard decks of cards.

The first game, called *Jack in 52*, is won by selecting a Jack of a specific suit from the deck. The probability of actually doing this is, of course, 1 in 52 (= 0.0192). Gamblers may place a bet of $1, $2, or $4 on this game. If they win, the payouts are $12 for a $1 bet, $24.55 for a $2 bet, and $49 for a $4 bet.

The second game, called *Red Face in 52*, is won by selecting from the deck a red face card (i.e., red Jack, red Queen, or red King). The probability of winning is 6 in 52 (= 0.1154). Again, bets may be placed in denominations of $1, $2, and $4. Payouts are $8.10, $16.35, and $32.50, respectively.

The third game, called *Face in 52*, is won by selecting one of the 12 face cards from the deck. The probability of winning is 12 in 52 (= 0.2308). Payouts are $4, $8.15, and $16 for $1, $2, and $4 bets.

The last game, called *Red in 52*, is won by selecting a red card from the deck. The probability of winning is 26 in 52 (= 0.50). Payouts are $1.80, $3.80, and $7.50 for $1, $2, and $4 bets.

Given that they can calculate the expected return (or, more appropriately, loss) for each type of game and level of wager, they have decided to see if they can minimize their total expected loss by planning their evening using LP. For example, the expected return from a $1 bet in the game *Jack in 52* is equal to $0.2308 (= $12 × 1/52 + $0 × 51/52). Since the amount bet is $1, the expected loss is equal to $0.7692 (= $1 − $0.2308). All other expected losses can be calculated in a similar manner.

They want to appear to be sociable and not as if they are trying to lose as little as possible. Therefore, they will place at least 20 bets (of any value) on each of the four games. Further, they will spend at least $26 on $1 bets, at least $50 on $2 bets, and at least $72 on $4 bets. They will bet no more than (and no less than) the agreed-upon $300. What should be their gambling plan, and what is their expected loss for the evening?

3-17 A hospital emergency room is open 24 hours a day. Nurses report for duty at 1 A.M., 5 A.M., 9 A.M., 1 P.M., 5 P.M., or 9 P.M., and each works an 8-hour shift. Nurses are paid the same, regardless of the shift they work. The following table shows the minimum number of nurses needed during the six periods into which the day is divided. How should the hospital schedule the nurses so that the total staff required for one day's operation is minimized?

SHIFT	TIME	NURSES NEEDED
1	1–5 A.M.	4
2	5–9 A.M.	13
3	9 A.M.–1 P.M.	17
4	1–5 P.M.	10
5	5–9 P.M.	12
6	9 P.M.–1 A.M.	5

3-18 A nursing home employs attendants who are needed around the clock. Each attendant is paid the same, regardless of when his or her shift begins. Each shift is 8 consecutive hours. Shifts begin at 6 A.M., 10 A.M., 2 P.M., 6 P.M., 10 P.M., and 2 A.M. The following table shows the nursing home's requirements for the numbers of attendants to be on duty during specific time periods.

SHIFT	TIME	NUMBER OF ATTENDANTS
A	2–6 A.M.	8
B	6–10 A.M.	27
C	10 A.M.–2 P.M.	12
D	2–6 P.M.	23
E	6–10 P.M.	29
F	10 P.M.–2 A.M.	23

(a) What is the minimum number of attendants needed to satisfy the nursing home's requirements?

(b) The nursing home would like to use the same number of attendants determined in part (a) but would now like to minimize the total salary paid. Attendants are paid $16 per hour during 8 A.M.–8 P.M., and a 25% premium per hour during 8 P.M.–8 A.M. How should the attendants now be scheduled?

3-19 A hospital is moving from 8-hour shifts for its lab techs to 12-hour shifts. Instead of working five 8-hour days, the lab techs would work three days on and four days off in the first week followed by four days on and three days off in the second week, for a total of 84 hours every two weeks.

Because the peak demand times in the hospital appear to be between 5 A.M. and 7 A.M. and between 5 P.M. and 7 P.M., four 12-hour shifts will be arranged according to the table at the top of p. 109.

SHIFTS	WORK TIMES	PAY RATE/ WEEK
A and A (alt)	5 A.M.–5 P.M.	$756
B and B (alt)	7 A.M.–7 P.M.	$840
C and C (alt)	5 P.M.–5 A.M.	$882
D and D (alt)	7 P.M.–7 A.M.	$924

The shift pay differentials are based on the most and least desirable times to begin and end work. In any one week, techs on shift A might work Sunday through Tuesday, while techs on shift A (alt) would work at the same times but on Wednesday through Saturday. In the following week, techs on shift A would work Sunday through Wednesday, while techs on shift A (alt) would work the corresponding Thursday through Saturday. Therefore, the same number of techs would be scheduled for shift A as for shift A (alt).

The requirements for lab techs during the 24-hour day are shown in the following table. What is the most economical schedule for the lab techs?

	5 A.M.– 7 A.M.	7 A.M.– 5 P.M.	5 P.M.– 7 P.M.	7 P.M.– 5 A.M.
Lab techs needed	12	8	14	10

3-20 An airline with operations in San Diego, California, must staff its ticket counters inside the airport. Ticket attendants work 6-hour shifts at the counter. There are two types of agents: those who speak English as a first language and those who are fully bilingual (English and Spanish). The requirements for the number of agents depend on the numbers of people expected to pass through the airline's ticket counters during various hours. The airline believes that the need for agents between the hours of 6 A.M. and 9 P.M. are as follows:

	6 A.M.– 9 A.M.	9 A.M.– NOON	NOON– 3 P.M.	3 P.M.– 6 P.M.	6 P.M.– 9 P.M.
Agents needed	12	20	16	24	12

Agents begin work either at 6 A.M., 9 A.M., noon, or 3 P.M. The shifts are designated as shifts A, B, C, and D, respectively. It is the policy of the airline that at least half of the agents needed in any time period will speak English as the first language. Further, at least one-quarter of the agents needed in any time period should be fully bilingual.

(a) How many and what type of agents should be hired for each shift to meet the language and staffing requirements for the airline, so that the total number of agents is minimized?

(b) What is the optimal hiring plan from a cost perspective if English-speaking agents are paid $25 per hour and bilingual agents are paid $29 per hour? Does the total number of agents needed change from that computed in part (a)?

3-21 A small trucking company is determining the composition of its next trucking job. The load master has his choice of seven different types of cargo, which may be loaded in full or in part. The specifications of the cargo types are shown in the following table. The goal is to maximize the amount of freight, in terms of dollars, for the trip. The truck can hold up to 900 pounds of cargo in a 2,500-cubic-foot space. What cargo should be loaded, and what will be the total freight charged?

CARGO TYPE	FREIGHT PER POUND	VOLUME PER POUND (CU. FT.)	POUNDS AVAILABLE
A	$8.00	3.0	210
B	$6.00	2.7	150
C	$3.50	6.3	90
D	$5.75	8.4	120
E	$9.50	5.5	130
F	$5.25	4.9	340
G	$8.60	3.1	250

3-22 The load master for a freighter wants to determine the mix of cargo to be carried on the next trip. The ship's volume limit for cargo is 100,000 cubic meters, and its weight capacity is 2,310 tons. The master has five different types of cargo from which to select and wishes to maximize the value of the selected shipment. However, to make sure that none of his customers are ignored, the load master would like to make sure that at least 20% of each cargo's available weight is selected. The specifications for the five cargoes are shown in the following table.

CARGO TYPE	TONS AVAILABLE	VALUE PER TON	VOLUME PER TON (CU. M.)
A	970	$1,350	26
B	850	$1,675	54
C	1,900	$1,145	28
D	2,300	$ 850	45
E	3,600	$1,340	37

3-23 A cargo transport plane is to be loaded to maximize the revenue from the load carried. The plane may carry any combination and any amount of cargoes A, B, and C. The relevant values for these cargoes are shown in the table at the top of p. 110.

CARGO TYPE	TONS AVAILABLE	REVENUE PER TON	VOLUME PER TON (CU. FT.)
A	10	$700	2,000
B	12	$725	3,500
C	17	$685	3,000

The plane can carry as many as 32 tons of cargo. The plane is subdivided into compartments, and there are weight and volume limitations for each compartment. It is critical for safety reasons that the weight ratios be strictly observed. The requirements for cargo distribution are shown in the following table.

COMPARTMENT	MAXIMUM VOLUME (CU. FT.)	COMPARTMENT WEIGHT/TOTAL WEIGHT RATIO
Right fore	16,000	Must equal 18% of total weight loaded
Right center	20,000	Must equal 25% of total weight loaded
Right aft	14,000	Must equal 7% of total weight loaded
Left fore	10,000	Must equal 18% of total weight loaded
Left center	20,000	Must equal 25% of total weight loaded
Left aft	12,000	Must equal 7% of total weight loaded

Which cargoes should be carried, and how should they be allocated to the various compartments?

3-24 The owner of a private freighter is trying to decide which cargo he should carry on his next trip. He has two choices of cargo, which he can agree to carry in any combination. He may carry up to 15 tons of cargo A, which takes up 675 cubic feet per ton and earns revenue of $85 per ton. Or, he may carry up to 54 tons of cargo B, with a volume of 450 cubic feet per ton and revenue of $79 per ton.

The freighter is divided into two holds, starboard and port. The starboard hold has a volume of 14,000 cubic feet and a weight capacity of 26 tons. The port hold has a volume of 15,400 cubic feet and a weight capacity of 32 tons. For steering reasons, it is necessary that the weight be distributed equally between the two sides of the freighter. However, the freighter engines and captain's bridge, which together weight 6 tons, are on the starboard side of the freighter. This means that the port side is usually loaded with 6 tons more cargo to equalize the weight. The owner may carry any combination of the two cargoes in the same hold without a problem. How should this freighter be loaded to maximize total revenue?

3-25 A farmer is making plans for next year's crop. He is considering planting corn, tomatoes, potatoes and okra. The data he has collected, along with the availability of resources, are shown in the table at the bottom of this page. He can plant as many as 60 acres of land. Determine the best mix of crops to maximize the farm's revenue.

3-26 The farmer in Problem 3-25 has an opportunity to take over the neighboring 80-acre farm. If he acquires this farm, he will be able to increase the amounts of time available to 1,600 hours for planting, 825 hours for tending, and 1,400 hours for harvesting. Between the two farms, there are 510 units of water and 6,000 pounds of fertilizer available. However, the neighboring farm has not been cultivated in a while. Therefore, each acre of this farm will take an additional 4 hours to plant and an additional 2 hours to tend. Because of the condition of the new farm, the farmer expects the yields per acre planted there to be only 46 bushels, 37 bushels, 42 bushels, and 45 bushels, respectively, for corn, tomato, potato, and okra. In order to make sure that both farms are used effectively, the farmer would like at least 80% of each farm's acreage to be planted. What is the best combination of crops to plant at each farm in order to maximize revenue?

3-27 A family farming concern owns five parcels of farmland broken into a southeast sector, north sector, northwest sector, west sector, and southwest sector. The farm concern is involved primarily in growing wheat, alfalfa, and barley crops and is currently preparing the production plan for next year.

Table for Problem 3-25

CROP	YIELD (BUSHELS/ACRE)	REVENUE/ BUSHEL	PLANTING (HOURS/ACRE)	TENDING (HOURS/ACRE)	HARVEST (HOURS/ACRE)	WATER (UNITS/ACRE)	FERTILIZER (POUNDS/ACRE)
Corn	50	$55	10	2	6	2.5	50
Tomato	40	$85	15	8	20	3.0	60
Potato	46	$57	12	2	9	2.0	45
Okra	48	$52	18	12	20	3.0	35
Available			775	550	775	300	2,500

The Water Authority has just announced its yearly water allotment, with this farm receiving 7,500 acre-feet. Each parcel can tolerate only a certain amount of irrigation per growing season, as specified in the following table.

PARCEL	AREA (ACRES)	WATER IRRIGATION LIMIT (ACRE-FEET)
Southeast	2,000	3,200
North	2,300	3,400
Northwest	600	800
West	1,100	500
Southwest	500	600

Each crop needs a minimum amount of water per acre, and there is a projected limit on sales of each crop, as noted in the following table.

CROP	MAXIMUM SALES	WATER NEEDED PER ACRE (ACRE-FEET)
Wheat	110,000 bushels	1.6
Alfalfa	1,800 tons	2.9
Barley	2,200 tons	3.5

Wheat can be sold at a net profit of \$2 per bushel, alfalfa at \$40 per ton, and barley at \$50 per ton. One acre of land yields an average of 1.5 tons of alfalfa and 2.2 tons of barley. The wheat yield is approximately 50 bushels per acre. What is the best planting plan for this farm?

3-28 A farmer has subdivided his land into three plots and wants to plant three crops in each plot: corn, rice, and soy. Plot sizes, crop acreage, profit per acre, and manure needed (pounds per acre) are given in the table at the bottom of this page.

The maximum acreage for each crop denotes the total acres of that crop that can be planted over all three plots. Currently there are 450,000 pounds of manure available. To ensure that the plots are used equitably, the farmer wants the same proportion of each plot to be under cultivation. (The proportion of each plot under cultivation must be the same for all three plots.) How much of each crop should be planted at each plot to maximize total profit?

3-29 A fuel cell manufacturer can hire union, non-union permanent, or temporary help. She has a contract to produce at the rate of 2,100 fuel cells per day and would like to achieve this at minimum cost. Union workers work 7 hours per day and can make up to 10 fuel cells per hour. Their wages and benefits cost the company \$15.00 and \$7.00 per hour, respectively. Union workers are assured that there will be no more than 80% of their number working in non-union permanent positions and that there will be no more than 20% of their number working in temporary positions.

Non-union permanent workers work 8 hours per day and can also make up to 10 fuel cells per hour. Their wages are the same as the union employees, but their benefits are worth only \$3.00 per hour. Temporary workers work 6 hours per day, can make up to 5 fuel cells per hour, and earn only \$10 per hour. They do not receive any benefits.

How many union, non-union, and temporary workers should be hired to minimize the cost to the manufacturer? What is the average cost of producing a fuel cell?

3-30 A chemical company wishes to mix three elements (E, F, and G) to create three alloys (X, Y, and Z). The costs of the elements are as shown in the following table.

ELEMENT	COST PER TON
E	\$3.00
F	\$4.00
G	\$3.50

To maintain the required quality for each alloy, it is necessary to specify certain maximum or minimum percentages of the elements. These are as shown in the following table.

ALLOY	SPECIFICATIONS	SELLING PRICE PER TON
X	No more than 30% of E, at least 40% of F, no more than 50% of G	\$5.50
Y	No more than 50% of E, at least 10% of F	\$4.00
Z	No more than 70% of E, at least 20% of G	\$6.00

Table for Problem 3-28

PLOT	ACREAGE
A	500
B	800
C	700

CROP	MAXIMUM ACREAGE	PROFIT PER ACRE	MANURE PER ACRE (POUNDS)
Corn	900	\$600	200
Wheat	700	\$450	300
Soy	1,000	\$300	150

Table for Problem 3-31

	INGREDIENT					
NUTRIENT	BEEF	PORK	CORN	LAMB	RICE	CHICKEN
Protein (%)	16.9	12.0	8.5	15.4	8.5	18.0
Fat (%)	26.0	4.1	3.8	6.3	3.8	17.9
Fiber (%)	29.0	8.3	2.7	2.4	2.7	28.8
Cost ($/lb)	0.52	0.49	0.20	0.40	0.17	0.39

The usage of each element is limited to 5,000 tons, and the total usage of all three elements is limited to 10,000 tons. Further, due to the relatively uncertain demand for alloy Z, the company would like to ensure that Z constitutes no more than 30% of the total quantity of the three alloys produced. Determine the mix of the three elements that will maximize profit under these conditions.

3-31 An animal feed company is developing a new puppy food. Their nutritionists have specified that the mixture must contain the following components by weight: at least 16% protein, 13% fat, and no more than 15% fiber. The percentages of each nutrient in the available ingredients, along with their cost per pound, are shown in the table at the top of this page.

What is the mixture that will have the minimum cost per pound and meet the stated nutritional requirements?

3-32 A boarding stable feeds and houses work horses used to pull tourist-filled carriages through the streets of a historic city. The stable owner wishes to strike a balance between a healthy nutritional standard for the horses and the daily cost of feed. This type of horse must consume exactly 5 pounds of feed per day. The feed mixes available are an oat product, a highly enriched grain, and a mineral product. Each of these mixes contains a predictable amount of five ingredients needed daily to keep the average horse healthy. The table at the bottom of this page shows these minimum requirements, units of each nutrient per pound of feed mix, and costs for the three mixes.

Formulate this problem and solve for the optimal daily mix of the three feeds.

3-33 Clint Hanks has decided to try a new diet that promises enhanced muscle tone if the daily intake of five essential nutrients is tightly controlled. After extensive research, Clint has determined that the recommended daily requirements of these nutrients for a person of his age, height, weight, and activity level are as follows: between 69 grams and 100 grams of protein, at least 700 milligrams of phosphorus, at least 420 milligrams of magnesium, between 1,000 milligrams and 1,750 milligrams of calcium, and at least 8 milligrams of iron. Given his limited finances, Clint has identified seven inexpensive food items that he can use to meet these requirements. The cost per serving for each food item and its contribution to each of the five nutrients are given in the table at the top of p. 113.

(a) Use LP to identify the lowest cost combination of food items that Clint should use for his diet.

(b) Would you characterize your solution in (a) as a well-balanced diet? Explain your answer.

3-34 A steel company is producing steel for a new contract. The contract specifies the information in the following table for the steel.

MATERIAL	MINIMUM	MAXIMUM
Manganese	2.10%	3.10%
Silicon	4.30%	6.30%
Carbon	1.05%	2.05%

The steel company mixes batches of eight different available materials to produce each ton of steel

Table for Problem 3-32

	FEED MIX			
NUTRIENT	OAT (UNITS/LB.)	GRAIN (UNITS/LB.)	MINERAL (UNITS/LB.)	NEEDED (UNITS/DAY)
A	2.0	3.0	1.0	6
B	0.5	1.0	0.5	2
C	3.0	5.0	6.0	9
D	1.0	1.5	2.0	8
E	0.5	0.5	1.5	5
Cost/lb.	$0.33	$0.44	$0.57	

Table for Problem 3-33

FOOD ITEM (SERVING SIZE)	PROTEIN (G)	PHOSPHORUS (MG)	MAGNESIUM (MG)	CALCIUM (MG)	IRON (MG)	COST PER SERVING ($)
Chicken Patty (0.25 pound)	17.82	250	29	23	1.14	0.50
Lasagna (300 grams)	24.53	223	56	303	2.52	0.58
2% Milk (1 cup)	8.05	224	27	293	0.05	0.42
Mixed Vegetables (1 cup)	5.21	93	40	46	1.49	0.24
Fruit Cocktail (1 cup)	1.01	26	11	15	0.62	0.37
Orange Juice (1 cup)	1.69	42	27	27	0.32	0.36
Oatmeal (1 packet)	4.19	136	46	142	10.55	0.18

according to the specification. The table at the bottom of this page details these materials.

Formulate and solve the LP model that will indicate how much of each of the eight materials should be blended into a 1-ton load of steel so that the company can meet the specifications under the contract while minimizing costs.

3-35 A meat packing house is creating a new variety of hot dog for the low-calorie, low-fat, low-cholesterol market. This new hot dog will be made of beef and pork, plus either chicken, turkey, or both. It will be marketed as a 2-ounce all-meat hot dog, with no fillers. Also, it will have no more than 6 grams of fat, no more than 27 grams of cholesterol, and no more than 100 calories. The cost per pound for beef, pork, chicken, and turkey, plus their calorie, fat, and cholesterol counts are shown in the following table.

	COST/ POUND	CALORIES/ POUND	FAT (G/LB.)	CHOLESTEROL (G/LB.)
Beef	$0.76	640	32.5	210
Pork	$0.82	1,055	54.0	205
Chicken	$0.64	780	25.6	220
Turkey	$0.58	528	6.4	172

The packer would like each 2-ounce hot dog to be at least 25% beef and at least 25% pork. What is the most economical combination of the four meats to make this hot dog?

3-36 A distributor imports olive oil from Spain and Italy in large casks. He then mixes these oils in different proportions to create three grades of olive oil that are sold domestically in the United States. The domestic grades include (a) commercial, which must be no more than 35% Italian; (b) virgin, which may be any mix of the two olive oils; and (c) extra virgin, which must be at least 55% Spanish. The cost to the distributor for Spanish olive oil is $6.50 per gallon. Italian olive oil costs him $5.75 per gallon. The weekly demand for the three types of olive oils is 700 gallons of commercial, 2,200 gallons of virgin, and 1,400 gallons of extra virgin. How should he blend the two olive oils to meet his demand most economically?

3-37 A paint company has two types of bases from which it blends two types of paints: Tuffcoat and Satinwear. Each base has a certain proportion of ingredients X, Y, and Z, as shown in the table at the top of p. 114, along with their costs.

Table for Problem 3-34

MATERIAL AVAILABLE	MANGANESE	SILICON	CARBON	POUNDS AVAILABLE	COST PER POUND
Alloy 1	70.0%	15.0%	3.0%	No limit	$0.12
Alloy 2	55.0%	30.0%	1.0%	300	$0.13
Alloy 3	12.0%	26.0%	0%	No limit	$0.15
Iron 1	1.0%	10.0%	3.0%	No limit	$0.09
Iron 2	5.0%	2.5%	0%	No limit	$0.07
Carbide 1	0%	24.0%	18.0%	50	$0.10
Carbide 2	0%	25.0%	20.0%	200	$0.12
Carbide 3	0%	23.0%	25.0%	100	$0.09

Table for Problem 3-37

	INGREDIENT X	INGREDIENT Y	INGREDIENT Z	COST/GALLON
Paint base A	25%	34%	10%	$4.50
Paint base B	35%	42%	15%	$6.50

The specifications for the two paints are shown in the following table.

TUFFCOAT	SATINWEAR
Must contain at least 33% ingredient X	Must contain at least 30% ingredient X
Must contain at least 35% ingredient Y	Must contain at least 38% ingredient Y
Must contain no more than 14% ingredient Z	Must contain no more than 13% ingredient Z
Demand = 1,600 gallons	Demand = 1,250 gallons

How should the two bases be blended to manufacture the two paints at a minimum cost? What is the cost per gallon for each paint?

3-38 The military has requested a new ready-to-eat meal (MRE) that will provide to troops in the field a very high-protein, low-carbohydrate instant canned breakfast. The can will contain 11 fluid ounces, or 325 mL, of the product. The design specifications are as follows: The drink should have at least 15 grams of protein, no more than 3 grams of fat, no more than 38 grams of carbohydrates, and no more than 310 mg of sodium. To make the drink, a food contractor plans to mix two ingredients it already makes, liquid A and liquid B, together with a new ingredient, liquid protein. The table at the bottom of this page describes the costs and the nutritional makeup of the three ingredients. Determine the least-cost mixture for the new MRE.

3-39 A commercial food for caged reptiles is made in 40-pound bags from five potential feeds. For labeling purposes, feed A must constitute at least 20% of each bag by weight, and each of feeds B to E must be at least 5% of the total weight. Further, feeds B and D must together constitute at least 30% by weight, and feeds B, C, and E together must be no more than 50% by weight. The costs per pound for feeds A to E are, respectively, $0.96, $0.85, $0.775, $0.45, and $0.375. How shall this reptile food be made, and what is the cost per bag?

3-40 A power company has just announced the August 1 opening of its second nuclear power-generation facility. The human resources department has been directed to determine how many nuclear technicians will need to be hired and trained over the remainder of the year. The plant currently employs 350 fully trained technicians and projects personnel needs as shown in the following table.

MONTH	HOURS NEEDED
August	40,000
September	45,000
October	35,000
November	50,000
December	45,000

By law, a reactor employee can actually work no more than 130 hours per month (Slightly over 1 hour per day is used for check-in and check-out, record keeping, and daily radiation health scans.) Company policy at the power company also dictates that layoffs are not acceptable in months when the nuclear power plant is overstaffed. So, if more trained employees are available than are needed in any month, each worker is still fully paid, even though he or she is not required to work the 130 hours.

Training new employees is an important and costly procedure. It takes one month of one-on-one classroom instruction before a new technician is permitted to work alone in the reactor facility. Therefore, trainees must be hired one month before they are actually needed. Each trainee teams up with a skilled nuclear technician and requires 90 hours of that employee's time, meaning that 90 hours less of the technician's time is available that month for actual reactor work.

Table for Problem 3-38

	COMPOSITION OF THE INGREDIENTS (PER LITER)				
	PROTEIN (G)	FAT (G)	CARBOHYDRATE (G)	SODIUM (MG)	COST/LITER
Liquid A	6	8	147	1770	$3.25
Liquid B	9	12	96	720	$4.50
Liquid protein	230	2	24	320	$28.00

Human resources department records indicate a turnover rate of trained technicians of 2% per month. In other words, 2% of the skilled technicians at the start of any month resign by the end of that month. A trained technician earns a monthly salary of $4,500, and trainees are paid $2,000 during their one month of instruction.

Formulate this staffing problem by using LP and solve it by using Excel.

3-41 A manufacturer of integrated circuits is planning production for the next four months. The forecast demand for the circuits is shown in the following table.

CIRCUIT	SEP	OCT	NOV	DEC
IC341	650	875	790	1,100
IC256	900	350	1,200	1,300

At the beginning of September, the warehouse is expected to be completely empty. There is room for no more than 1,800 integrated circuits to be stored. Holding costs for both types is $0.05 per unit per month. Because workers are given time off during the holidays, the manufacturer wants to have at least 800 IC341s and 850 IC256s already in the warehouse at the beginning of January.

Production costs are $1.25 per unit for IC341 and $1.35 per unit for IC256. Because demand for raw materials is rising, production costs are expected to rise by $0.05 per month through the end of the year.

Labor to make model IC341 is 0.45 hours per unit; making model IC256 takes 0.52 hours of labor. Management has agreed to schedule at least 1,000 hours per month of labor. As many as 200 extra hours per month are available to management at the same cost, except during the month of December, when only 100 extra hours are possible. What should be the production schedule for IC341 and IC256 for the four months?

3-42 A woman inherited $356,000. As she had no immediate need for the money at the time she inherited, she decided to invest some or all of it on January 1, 201*n*, with a goal of making the money grow to $500,000 by December 31, 201*n* + 5. She is considering the investments in the following table.

	RATE	MATURES
Fund A	7%	December 31 (at the end of one year)
Fund B	16%	December 31 (at the end of the second year after investment)
Fund C	24%	December 31 (at the end of the third year after investment)
Fund D	32%	December 31 (at the end of the fourth year after investment)

She wants to set up her investment strategy at the start of year 1. If she does not need to invest all of the inheritance to have $500,000 at the end of year 6, she will find another purpose for the remainder. She may choose to place a sum of money in any or all of the investments available at the start of year 1. From that point, however, all subsequent investments should come from the matured investments of previous years. To ensure that funds are spread over different investment choices, she does not want any single *new* investment in any year to be over $120,000. (Note that prior investments in a fund do not count toward this limit.)

How much money will she have to invest on January 1, 201*n*, to meet her goal of $500,000 at the end of the sixth year?

3-43 The Transportation Security Administration (TSA) at a large airport has 175 agents hired and trained for the month of January. Agents earn an average of $3,300 per month and work 160 hours per month. The projection is that 26,400 agent-hours will be required in February, 29,040 agent hours will be required in March, and 31,994 agent hours will be required in each of the months of April and May. Attrition during the month of January is anticipated to be 5%, so only 95% of the agents trained and working in January will be available for work in February. Efforts are being made to improve attrition: The TSA expects to lose only 4% of agents during February, 3% in March, and 2% in May. To ensure that enough agents will be available to meet the demand, new agents must be hired and trained. During the one-month training period, trainees are paid $2,600. Existing agents, who normally work 160 hours per month, are able to work only 80 hours during the months they are training new people. How many agents should be hired during the months of January to May?

3-44 A paper mill sells rolls of paper to newspapers, which usually place orders for rolls of different widths. The mill has just received a large order for 1.5 million feet of 4-foot-wide paper, 6 million feet of 9-foot-wide paper, and 3 million feet of 12-foot-wide paper. It produces rolls of two sizes: (1) 3,000 feet long and 14 feet wide, at a cost of $600 per roll, and (2) 3,000 feet long and 20 feet wide, at a cost of $1,100 per roll. Large cutting machines are then used to cut these rolls to rolls of desired widths.

(a) What should the paper mill do to satisfy this order at minimum cost? *Hint:* You need to first identify the different ways in which 14-foot-wide and 20-foot-wide rolls can be cut into 4-, 9-, and 12-foot-wide rolls.

(b) The paper mill is very concerned about the environment. Rather than determine the cheapest way of satisfying the current order, the firm

Table for Problem 3-45

	CUSTOMERS ($ PER UNIT SHIPPED)			
PLANT	SAVANNAH	MOBILE	ROANOKE	AVAILABLE
Columbia	$13	$42	$38	450
Greensboro	$25	$48	$17	290
Required	250	225	210	

would like to determine the least wasteful way (i.e., minimize the amount of paper wasted). What is the solution with this revised objective, and what is the new cost?

3-45 A company that manufactures products in two plants ships locally using its own transportation system, but it has orders that must be sent to customers too far away to be serviced by the local fleet. It therefore contracts with a middle-distance carrier to complete its shipping. The locations of the two manufacturing plants, amounts available at each plant to be shipped per week, locations of the three customers, their weekly requirements, and shipping costs ($ per unit) between each plant and customer are shown in the table at the top of this page.

What is the optimal shipping plan to satisfy the demand at the lowest total shipping cost?

3-46 A school district must determine which students from each of the four attendance zones will attend which of the three high schools. The north attendance zone is 8 miles from Central High School, 4 miles from Northwestern High School, and 16 miles from Southeastern High School. All of the distances (in miles) are shown in the following table.

ATTENDANCE ZONE	NORTH	SOUTH	EAST	WEST
Central High School	8	5	5	11
Northwestern High School	4	17	15	6
Southeastern High School	16	6	6	18
Number of Students	903	741	923	793

Each school can have as many as 1,200 students enrolled. The school district would like to make the allocation of students to schools that will minimize the number of miles necessary to transport the students.

3-47 The school district in Problem 3-46 wishes to impose an additional constraint on the problem: It wants to enroll the same number of students in each of the three schools. Solve for the revised allocation of students to schools that will minimize the number of miles necessary to transport the students.

Case Study

Chase Manhattan Bank

The workload in many areas of bank operations has the characteristics of a non-uniform distribution with respect to time of day. For example, at Chase Manhattan Bank in New York, the number of domestic money transfer requests received from customers, if plotted against time of day, would appear to have the shape of an inverted U curve, with the peak around 1 P.M. For efficient use of resources, the personnel available should, therefore, vary correspondingly. Figure 3.1 shows a typical workload curve and corresponding personnel requirements at different hours of the day.

A variable capacity can be achieved effectively by employing part-time personnel. Because part-timers are not entitled to all the fringe benefits, they are often more economical than full-time employees. Other considerations, however, may limit the extent to which part-time people can be hired in a given department. The problem is to find an optimum workforce schedule that would meet personnel requirements at any given time and also be economical.

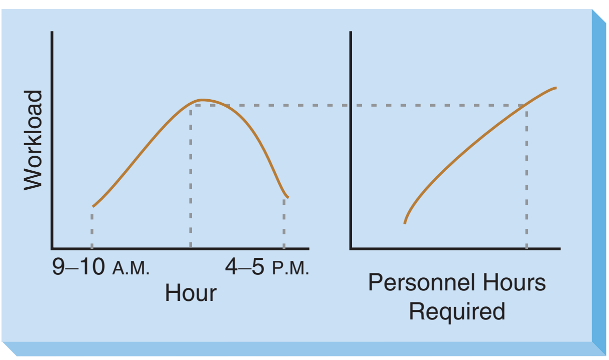


FIGURE 3.1 Figure for Case Study: Chase Manhattan Bank

Some of the factors affecting personnel assignment are listed here:

a. Full-time employees work for 8 hours per day, with a 1 hour break for lunch included.
b. Fifty percent of the full-timers go to lunch between 11 A.M. and noon, and the remaining 50% go between noon and 1 P.M.
c. Part-timers work for at least 4 continuous hours but no more than 7 continuous hours per day and are not allowed a lunch break.
d. By corporate policy, part-time personnel hours are limited to a maximum of 40% of the day's total requirement.
e. The shift starts at 9 A.M. and ends at 7 P.M. (i.e., overtime is limited to 2 hours). Any work left over at 7 P.M. is considered holdover for the next day.
f. A full-time employee is not allowed to work more than 1 hour of overtime per day. He or she is paid at the normal rate even for overtime hours—*not* at one and one-half times the normal rate typically applicable to overtime hours. Fringe benefits are not applied to overtime hours.

In addition, the following costs are pertinent:

a. The average normal rate per full-time personnel hour is $24.08.
b. The fringe benefit rate per full-time personnel hour is charged at 25% the normal rate.
c. The average rate per part-time personnel hour is $17.82.

The personnel hours required, by hour of day, are given in Table 3.17. The bank's goal is to achieve the minimum possible personnel cost subject to meeting or exceeding the hourly workforce requirements as well as the constraints on the workers listed earlier.

TABLE 3.17 Data for Chase Manhattan Bank

TIME PERIOD	HOURS REQUIRED
9–10 A.M.	14
10–11	25
11–12	26
12–1 P.M.	38
1–2	55
2–3	60
3–4	51
4–5	29
5–6	14
6–7	9

Discussion Questions

1. What is the minimum-cost schedule for the bank?
2. What are the limitations of the model used to answer question 1?

Source: Based on Shyam L. Moondra. "An L. P. Model for Work Force Scheduling for Banks," *Journal of Bank Research* (Winter 1976), 299–301.

Internet Case Studies

See the Companion Website for this textbook, at www.pearsonhighered.com/balakrishnan, for additional case studies.

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CHAPTER 4

Linear Programming Sensitivity Analysis

LEARNING OBJECTIVES

After completing this chapter, students will be able to:

1. Understand, using graphs, the impact of changes in objective function coefficients, right-hand-side values, and constraint coefficients on the optimal solution of a linear programming problem.
2. Generate Answer and Sensitivity Reports using Excel's Solver.
3. Interpret all parameters of these reports for maximization and minimization problems.
4. Analyze the impact of simultaneous changes in input data values using the 100% rule.
5. Analyze the impact of the addition of a new variable using the pricing-out strategy.

CHAPTER OUTLINE

4.1 Introduction

We have solved LP models under deterministic assumptions.

Optimal solutions to linear programming (LP) problems have thus far been found under what are called *deterministic* assumptions. This means that we assume complete certainty in the data and relationships of a problem—namely, prices are fixed, resources' availabilities are known, production time needed to make a unit are exactly set, and so on. That is, we assume that all the coefficients (constants) in the objective function and each of the constraints are fixed and do not change. But in most real-world situations, conditions are dynamic and changing. This could mean, for example, that just as we determine the optimal solution to an LP model that has the profit contribution for a given product set at $10 per unit, we find out that the profit contribution has changed to $9 per unit. What does this change mean for our solution? Is it no longer optimal?

Managers are often interested in studying the impact of changes in the values of input parameters.

In practice, such changes to input data values typically occur for two reasons. First, the value may have been estimated incorrectly. For example, a firm may realize that it has overestimated the selling price by $1, resulting in an incorrect profit contribution of $10 per unit, rather than $9 per unit. Or it may determine during a production run that it has only 175 pumps in inventory, rather than 200, as specified in the LP model. Second, management is often interested in getting quick answers to a series of what-if questions. For example, what if the profit contribution of a product decreases by 10%? What if less money is available for advertising? What if workers can each stay one hour longer every day at 1.5-times pay to provide increased production capacity? What if new technology will allow a product to be wired in one-third the time it used to take?

Why Do We Need to Study Sensitivity Analysis?

Sensitivity analysis, also known as *postoptimality analysis*, is a procedure that allows us to answer questions such as those posed above, using the current optimal solution itself, without having to resolve the LP model each time. Before we discuss this topic in more detail, let us first address a question that may arise commonly: Why do we need to study sensitivity analysis when we can use the computer to make the necessary changes to the model and quickly solve it again? The answer is as follows.

If the change in an input data value is certain, the easiest approach is to change it in the formulation and resolve the model.

If, in fact, we know that a change in an input data value is definite (e.g., we know *with certainty* that the profit contribution has decreased from $10 to $9 per unit), the easiest and logical course of action is to do just what the question suggests. That is, we should simply change the input data value in the formulation and solve the model again. Given the ease with which most real-world models can be solved using computers today, this approach should not be too difficult or time-consuming. Clearly, this same approach can be used even if we are making *definite* changes to more than one input data value at the same time.

In contrast, what if changes in input data values are just hypothetical, such as in the various what-if scenarios listed earlier? For example, assume that we are just considering lowering the selling price of a product but have not yet decided to what level it should be lowered. If we are considering 10 different selling price values, changing the input data value and resolving the LP model for every proposed value results in 10 separate models. If we expand this argument to consider 10 selling price levels each for two different products, we now have 100 ($= 10 \times 10$) LP models to solve. Clearly, this approach (i.e., changing and resolving the LP model) quickly becomes impractical when we have many input data values in a model and we are considering what-if multiple changes in each of their values.

Sensitivity analysis involves examining how sensitive the optimal solution is to changes in profits, resources, or other input parameters.

In such situations, the preferred approach is to formulate and solve a *single* LP model with a given set of input data values. However, after solving this model, we conduct a sensitivity analysis of the optimal solution to see just how *sensitive* it is to changes in each of these input data values. That is, for each input data value, we attempt to determine a *range of values* within which the current optimal solution will remain optimal. For example, if the current selling price for a product is $10 per unit, we identify the extent to which this value can change (both on the higher side and on the lower side) without affecting the optimality of the current solution. We can obtain this information, as we shall see, from the current solution itself, without resolving the LP model each time.

As we did previously with LP formulations and solutions, we first study LP sensitivity analysis using a two-variable product mix problem. We recognize here again that we are unlikely

IN ACTION The Right Stuff at NASA

There are many areas at NASA where decision modeling tools such as linear programming have been applied successfully. With the culture of the U.S. space program changing because of an increasing pressure to develop missions under rigid schedule and budget constraints, NASA has to work in an environment of faster–better–cheaper. After its highly publicized Mars failure, NASA addresses the issue of scarce resources with a technique called *sensitivity analysis* and a measure of marginal costs called the *shadow price*.

For example, at the margin, money is sometimes spent on tests that are not justified by the value of information. In other cases, more funds would yield risk-reduction benefits that would justify the costs and greatly improve the scientific benefit of a space mission. The shadow prices of the resource constraints in such a case provide valuable insights into the cost–benefit trade-off.

When NASA looks at a project that is critical to the success of new missions, the costs of losing the project will include marginal values of the delays and loss of data incurred by these future missions. Linear programming doesn't just provide optimal solutions; it provides the ability to conduct sensitivity analysis on these solutions as well.

Source: From M. E. Pate-Cornell and R. L. Dillon. "The Right Stuff," *OR/MS Today* 27, 1 (February 2000): 36–39, reprinted with permission.

Excel's Solver can be used to generate Sensitivity Reports.

to encounter two-variable problems in real-world situations. Nevertheless, a big advantage of studying such models is that we can demonstrate the concepts of sensitivity analysis using a graphical approach. This experience will be invaluable in helping understand the various issues in sensitivity analysis even for larger problems. For these larger problems, because we cannot view them graphically, we will rely on Excel's Solver to generate a Sensitivity Report. We discuss three separate Solver Sensitivity Reports in this chapter: (1) a report for the two-variable product mix problem that we also first analyze graphically, (2) a report for a larger problem (i.e., more than two variables) with a maximization objective function, and (3) a report for a larger problem with a minimization objective function. The two larger problems allow us to illustrate fully the various types of information we can obtain by using sensitivity analysis.

We will first study the impact of only one change at a time.

We will initially study sensitivity analysis by varying only one input data value at a time. Later, we will expand our discussion to include simultaneous changes in several input data values.

4.2 Sensitivity Analysis Using Graphs

To analyze LP sensitivity analysis by using graphs, let us revisit the Flair Furniture problem that we first used in Chapter 2 to introduce LP formulation and solution. Our motivation for using the same problem here is that you are hopefully already familiar with that problem and its graphical solution. Nevertheless, you might want to briefly review sections 2.3 and 2.4 in Chapter 2 before proceeding further.

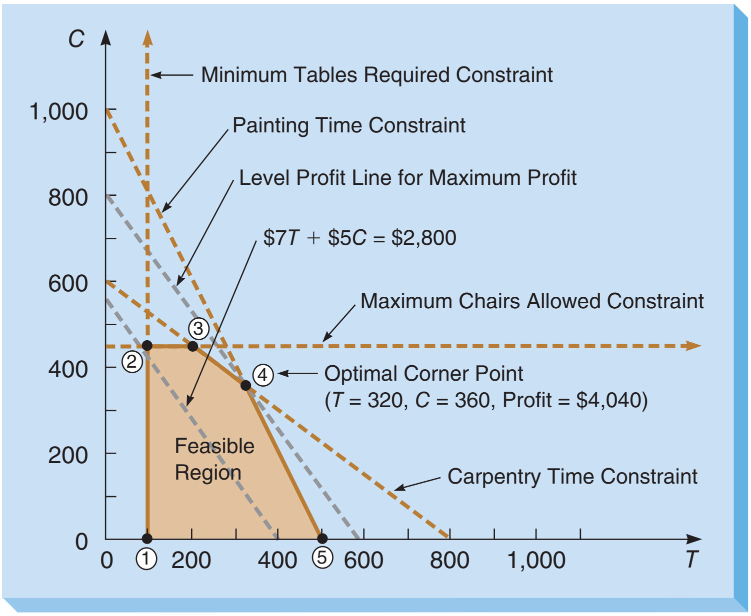
Recall that the Flair Furniture Company problem involved two products: tables and chairs. The constraints dealt with the hours available in the carpentry and painting departments, production limits on chairs, and the minimum production level on tables. If we let T denote the number of tables to make and C denote the number of chairs to make, we can formulate the following LP problem to determine the best product mix:

$$\text{Maximize profit} = \$7T + \$5C$$

subject to the constraints

$$\begin{aligned}
3T + 4C &\le 2{,}400 && \text{(carpentry time)}\\
2T + 1C &\le 1{,}000 && \text{(painting time)}\\
C &\le 450 && \text{(maximum chairs allowed)}\\
T &\ge 100 && \text{(minimum tables required)}\\
T, C &\ge 0 && \text{(nonnegativity)}
\end{aligned}$$

FIGURE 4.1
Optimal Corner Point Solution for Flair Furniture



The solution to this problem is illustrated graphically in Figure 4.1 (which is essentially the same information as in Figure 2.6 on page 31 in Chapter 2). Recall from Chapter 2 that we can use the level profit lines method to identify the optimal corner point solution. (The level profit line for a profit value of $2,800 is shown in Figure 4.1.) It is easy to see that Flair's optimal solution is at corner point ④. At this corner point, the optimal solution is to produce 320 tables and 360 chairs, for a profit of $4,040.

Types of Sensitivity Analysis

In the preceding LP formulation, note that there are three types of input parameter values:

1. ***Objective function coefficient (OFC).*** The **OFCs** are the coefficients for the decision variables in the objective function (such as the $7 and $5 for T and C, respectively, in Flair's model). In many business-oriented LP models, OFCs typically represent unit profits or costs, and they are measured in monetary units such as dollars, euros, and rupees.

   Are OFCs likely to have any uncertainty in their values? Clearly, the answer is yes because in many real-world situations, selling and cost prices are seldom likely to be static or fixed. For this reason, we will study how the optimal solution may be affected by changes in OFC values.
2. ***Right-hand-side (RHS) value of a constraint.*** The **RHS values** are constants, such as the 2,400 and 1,000 in Flair's model, that typically appear on the RHS of a constraint (i.e., to the right of the equality or inequality sign). For $\leq$ constraints, they typically represent the amount available of a resource, and for $\geq$ constraints, they typically represent the minimum level of satisfaction needed.

   Are these types of input data subject to uncertainty in practice? Here again, the answer is a clear yes. In many practical situations, companies may find that their resource availability has changed due to, for example, miscounted inventory, broken-down machines, absent labor, etc. For this reason, we will study how the optimal solution may be affected by changes in RHS values.
3. ***Constraint coefficient.*** The constraint coefficients are the coefficients for the decision variables in a model's constraints (such as the 3 and 4 in the carpentry constraint in Flair's model). In many problems, these represent design issues with regard to the decision variables. For example, needing three hours of carpentry per table is a product design issue that has probably been specified by design engineers.

   Although we could think of specific situations where these types of input parameters could also be subject to uncertainty in their values, such changes are less likely here than in OFC and RHS values. For this reason, we do not usually study the impact of changes in constraint coefficient values on the optimal solution.

Most computer-based LP software packages, including Excel's Solver, provide Sensitivity Reports only for analyzing the effect of changes in OFC and RHS values.

Impact of Changes in an Objective Function Coefficient

We examine changes in OFCs first.

When the value of an OFC changes, the feasible solution region remains the same (because it depends only on the constraints). That is, we have the same set of corner points, and their locations do not change. All that changes is the slope of the level profit (or cost) line.

Let us consider the impact of changes in the profit contribution of tables (T). First, what if the demand for tables becomes so high that the profit contribution can be raised from \$7 to \$8 per table? Is corner point ④ still the optimal solution? The answer is definitely yes, as shown in Figure 4.2. In this case, the slope of the level profit line accentuates the optimality of the solution at corner point ④. However, even though the decision variable values did not change, the new optimal objective function value (i.e., the profit) does change and is now \$4,360 (= \$8 × 320 + \$5 × 360).

In a similar fashion, let us analyze what happens if the demand for tables forces us to reduce the profit contribution from \$7 to \$6 per table. Here again, we see from the level profit line in Figure 4.2 that corner point ④ continues to remain the optimal solution, and the production plan does not change. The optimal profit, however, is now only \$3,720 (= \$6 × 320 + \$5 × 360).

If the OFC changes too much, a new corner point could become optimal.

On the other hand, what if a table's profit contribution can be raised all the way to \$11 per table? In such a case, the level profit line, shown in Figure 4.3, indicates that the optimal solution is now at corner point ⑤, instead of at corner point ④. The new solution is to make 500 tables and 0 chairs, for a profit of \$5,500. That is, tables are now so profitable compared to chairs that we should devote all our resources to making only tables.

Likewise, what if a table's profit contribution was highly overestimated and should only have been \$3 per table? In this case also, the slope of the level profit line changes enough to cause a new corner point ③ to become optimal (as shown in Figure 4.3). That is, tables have now become relatively unattractive compared to chairs, and so we will make fewer tables and more chairs. In fact, the only reason we even make any tables in this case is because we are explicitly constrained in the problem to make at least 100 tables, and from making more than 450 chairs. At corner point ③, the solution is to make 200 tables and 450 chairs, for a profit of \$2,850 (= \$3 × 200 + \$5 × 450).

There is a range for each OFC over which the current solution remains optimal.

From the preceding discussion regarding the OFC for a table, it is apparent that there is a range of possible values for this OFC for which the *current* optimal corner point solution remains optimal. Any change in the OFC value beyond this range (either on the higher end or the lower end) causes a *new* corner point to become the optimal solution. Clearly, we can repeat the same discussion with regard to the OFC for chairs.

FIGURE 4.2
Small Changes in Profit Contribution of Tables

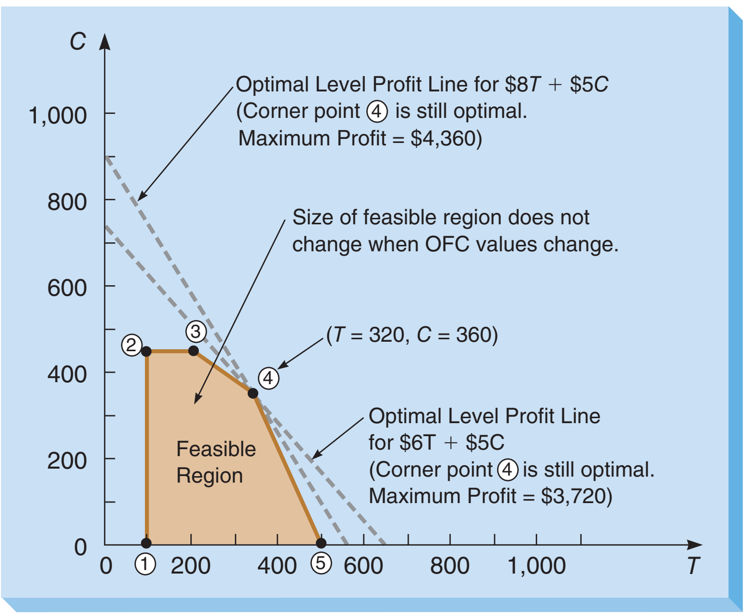
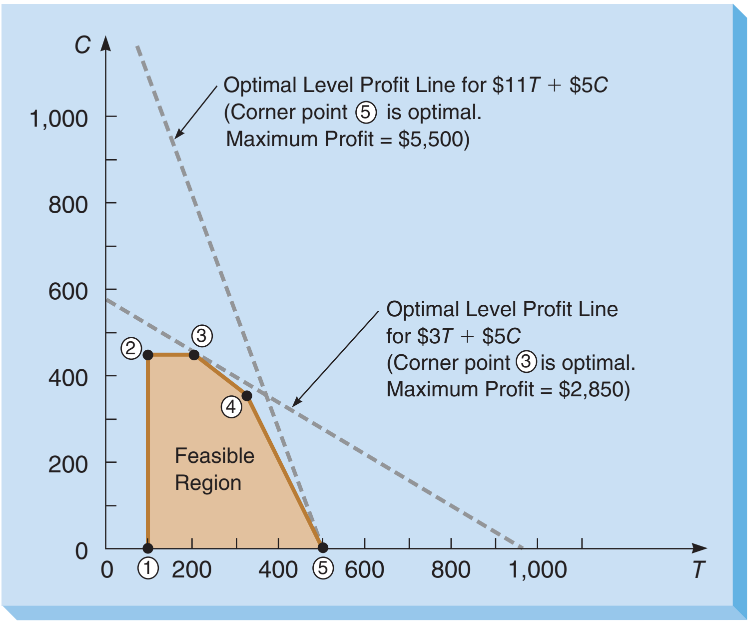


FIGURE 4.3
Larger Changes in Profit Contribution of Tables



It is algebraically possible to use the graphical solution procedure to determine the allowable range for each OFC within which the current optimal solution remains optimal. However, we use the information provided in the Solver Sensitivity Report to discuss this issue further in the next section.

Changes in OFC values do not affect the size of the feasible region.

Again, whenever changes occur in OFC values, the feasible region of the problem (which depends only on the constraints) does not change. Therefore, there is no change in the physical location of each corner point. To summarize, only two things can occur due to a change in an OFC: (1) If the current optimal corner point continues to remain optimal, the decision variable values do not change, even though the objective function value may change; and (2) if the current corner point is no longer optimal, the values of the decision variables change, as does the objective function value.

Impact of Changes in a Constraint's Right-Hand-Side Value

Changes in RHS values could affect the size of the feasible region.

Unlike changes in OFC values, a change in the RHS value of a nonredundant constraint results in a change in the size of the feasible region.[1] Hence, one or more corner points may physically shift to new locations. Recall from Chapter 2 that at the optimal solution, constraints can either be binding or nonbinding. Binding constraints intersect at the optimal corner point and are, hence, exactly satisfied at the optimal solution. Nonbinding constraints have a nonzero slack (for ≤ constraints) or surplus (for ≥ constraints) value at the optimal solution. Let us analyze impacts of changes in RHS values for binding and nonbinding constraints separately.

IMPACT OF CHANGE IN RHS VALUE OF A BINDING CONSTRAINT From Figure 4.1, we know that the two binding constraints in Flair's problem are the carpentry and painting hours. Let us analyze, for example, potential changes in the painting hours available. Flair currently projects an availability of 1,000 hours, all of which will be needed by the current production plan.

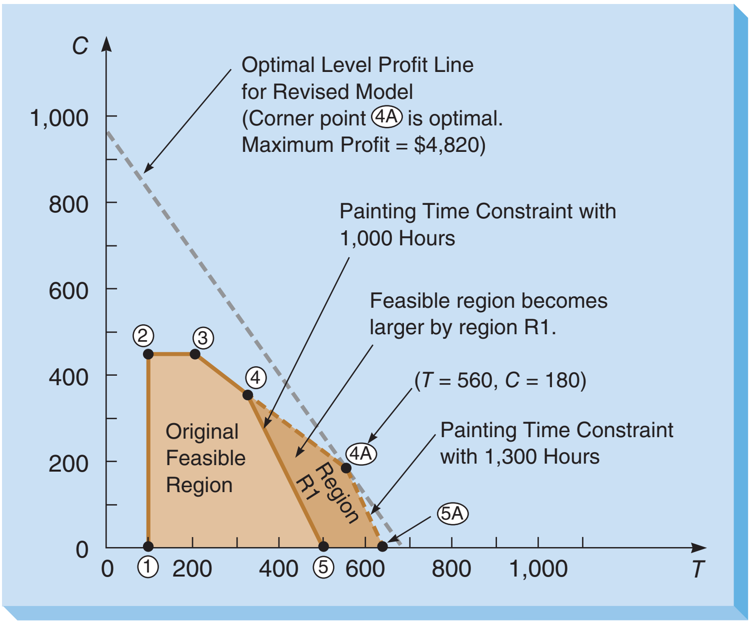
Let us first analyze the impact if this value is increased. What happens if, for example, the painting time availability can be increased by 300 hours (to 1,300 hours) by adding an extra painter? Figure 4.4 shows the revised graph for Flair's problem under this scenario.

The location of the optimal corner point changes if the RHS of a binding constraint changes.

The first point to note is that because the painting constraint is a binding ≤ constraint, any increase in its RHS value causes the feasible region to become larger, as shown by the region marked R1 in Figure 4.4. As a consequence of this increase in the size of the feasible region, the locations of corner points ④ and ⑤ shift to new locations—4A and 5A, respectively. However, the level profit lines approach (shown in Figure 4.4) indicates that the intersection of the carpentry and painting constraints (i.e., corner point 4A) is still the optimal solution. That is, the

[1] Recall from section 2.6 in Chapter 2 that a *redundant* constraint does not affect the feasible region in any way.

FIGURE 4.4
Increase in Availability of Painting Hours to 1,300 Hours

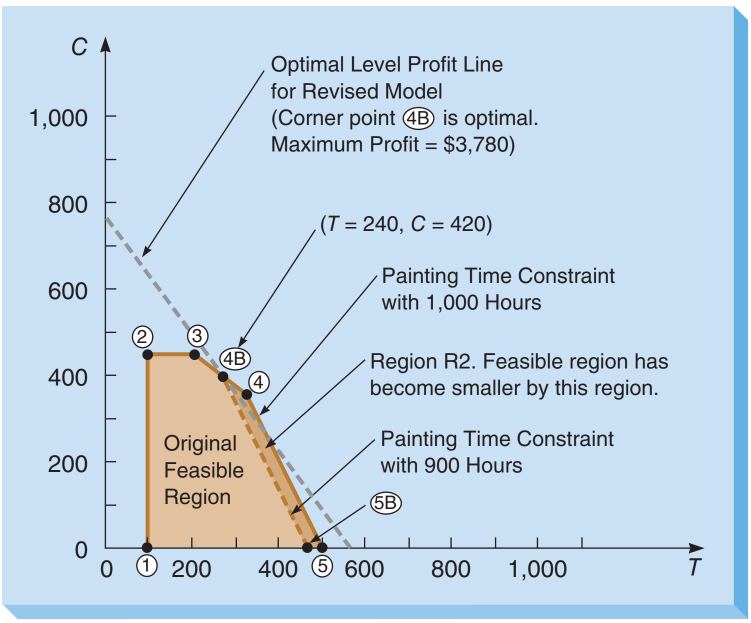


"same" corner point (in the sense that the same two constraints intersect at this point) is still optimal. But it now has a new location and, hence, there are new values for T, C, and profit. The values at corner point 4A can be computed to be $T = 560$ and $C = 180$, for a profit of \$4,820. This implies that if Flair is able to obtain an additional 300 hours of painting time, it can increase profit by \$780 (from \$4,040 to \$4,820) by revising the production plan. This profit increase of \$780 for 300 additional hours of painting time translates to a profit increase of \$2.60 per additional hour of painting time.

The feasible region becomes smaller if the RHS value of a binding ≤ constraint is decreased.

Next, let us analyze the impact if the painting time availability is decreased. What happens if, for example, this value is only 900 hours instead of 1,000 hours? The revised graph, shown in Figure 4.5, indicates that this decrease in the RHS value of a binding ≤ constraint shrinks the size of the feasible region (as shown by the region marked R2 in Figure 4.5). Here again, the locations of corner points 4 and 5 have shifted to new locations, 4B and 5B, respectively. However, as before, the level profit lines approach indicates that the "same" corner point (i.e., intersection of the carpentry and painting constraints, point 4B) is still optimal. The values of the decision variables and the resulting profit at corner point 4B can be computed to be

FIGURE 4.5
Decrease in Availability of Painting Hours to 900 Hours



$T = 240$ and $C = 420$, for a profit of \$3,780. That is, the loss of 100 hours of painting time causes Flair to lose \$260 in profit (from \$4,040 to \$3,780). This translates to a decrease in profit of \$2.60 per hour of painting time lost.

The shadow price *is the change in objective function value for a one-unit increase in a constraint's RHS value.*

Observe that the profit increases by \$2.60 per each additional hour of painting time gained, and it decreases by the *same* \$2.60 per each hour of painting time lost from the current level. This value, known as the **shadow price**, is an important concept in LP models. The shadow price of a constraint can be defined as the change in the optimal objective function value for a one-unit increase in the RHS value of that constraint. In the case of painting time, the shadow price is \$2.60; this implies that each hour of painting time (with respect to the current availability) affects Flair's profit by \$2.60. Because painting time is a binding $\leq$ constraint, each additional hour obtained increases profit by \$2.60, while each hour lost decreases profit by \$2.60.

The shadow price is valid only for a certain range of change in a constraint's RHS value.

Is this shadow price of \$2.60 valid for any level of change in the painting time availability? That is, for example, can Flair keep obtaining additional painting time and expect its profit to keep increasing endlessly by \$2.60 for each hour obtained? Clearly, this cannot be true, and we illustrate the reason for this in the following section.

Increasing the RHS of a $\leq$ constraint endlessly will eventually make it a redundant constraint.

VALIDITY RANGE FOR THE SHADOW PRICE Consider, for example, what happens if Flair can increase the painting time availability even further, to 1,700 hours. Under this scenario, as shown in Figure 4.6, the feasible region increases by the region marked R3. However, due to the presence of the nonnegativity constraint $C \geq 0$, the corner point defined by the intersection of the carpentry and painting constraints is no longer feasible. In fact, the painting constraint has now become a redundant constraint. Obviously, in such a case, the optimal solution has shifted to a new corner point. The level profit lines approach indicates that the optimal solution is now at corner point 5C ($T = 800$, $C = 0$, profit $=$ \$5,600). Note that this translates to a profit increase of \$1,560 ($=$ \$5,600 $-$ \$4,040) for 700 additional hours, or \$2.23 per hour, which is different from the shadow price of \$2.60. That is, the shadow price of \$2.60 is not valid for an increase of 700 hours in the painting time availability.

Decreasing the RHS of a $\leq$ constraint endlessly will eventually make some other constraint a redundant constraint.

What happens if the painting time availability is decreased all the way down to 700 hours? Here again, as shown in Figure 4.7, the intersection point of the carpentry and painting constraints is no longer even feasible. The carpentry constraint is now redundant, and the optimal solution has switched to a new corner point given by corner point 3A ($T = 125$, $C = 450$, profit $=$ \$3,125). This translates to a profit decrease of \$915 ($=$ \$4,040 $-$ \$3,125) for a decrease of 300 hours, or \$3.05 per hour, which is again different from the shadow price of \$2.60. That is, the shadow price of \$2.60 is not valid for a decrease of 300 hours in the painting time availability.

There is a range of values for each RHS for which the current corner points exist.

The preceding discussion based on Figures 4.4 to 4.7 shows that for a certain range of change in the RHS value of a binding constraint, the "same" corner point will continue to remain optimal.

FIGURE 4.6
Increase in Availability of Painting Hours to 1,700 Hours

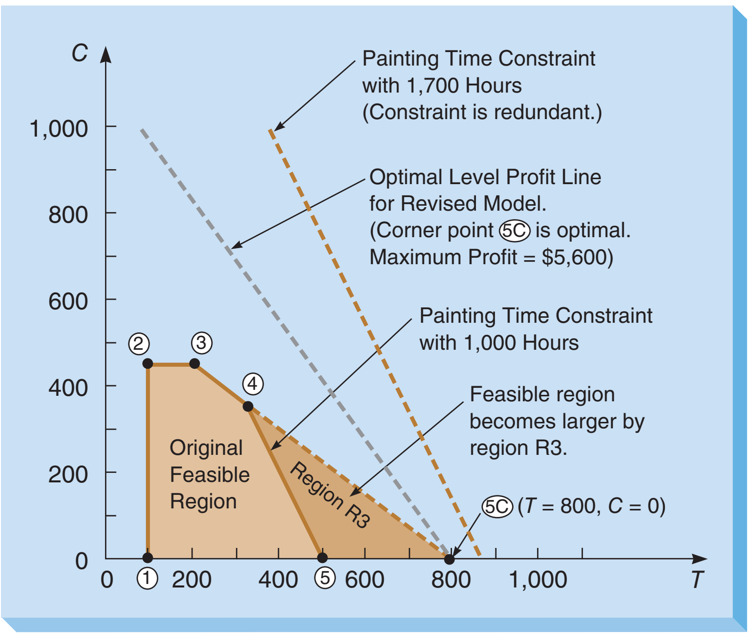
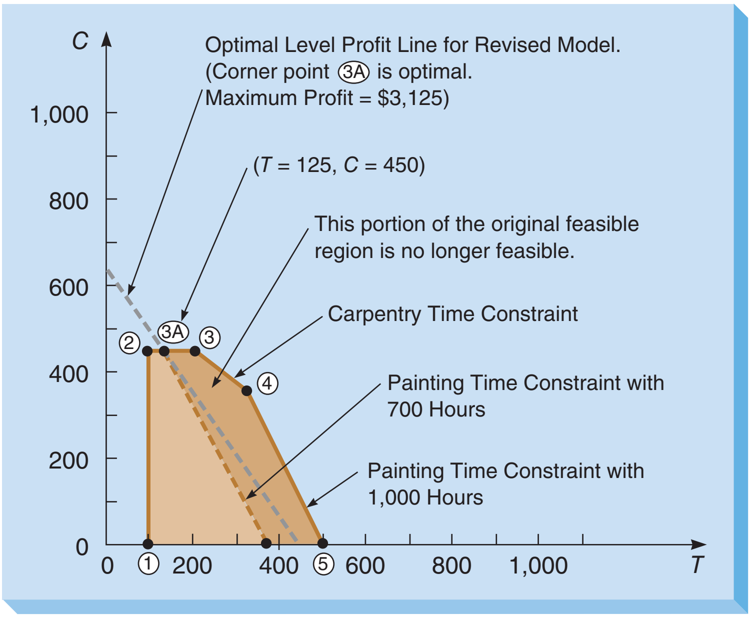


FIGURE 4.7
Decrease in Availability of Painting Hours to 700 Hours



That is, the constraints that are currently binding at the optimal solution will continue to remain the binding constraints. The location of this optimal corner point will, however, change, depending on the change in the RHS value. In fact, it turns out that as long as this corner point exists in the feasible region, it will continue to remain optimal. In Flair's case, this means that the corner point where the carpentry and painting constraints intersect will remain the optimal solution *as long as it exists in the feasible region*. Also, the shadow price of \$2.60 measures the impact on profit for a unit change in painting time availability as long as this corner point continues to exist in the feasible region. Once this RHS value changes to such an extent that the current binding constraints no longer intersect in the feasible region, the shadow price of \$2.60 is no longer valid and changes to a different value. It is algebraically possible to use the graphical solution to determine the RHS range within which the current optimal corner point continues to exist, albeit at a new location. We will, however, use the information provided in the Solver Sensitivity Report to further discuss this issue in a subsequent section.

A similar analysis can be conducted with the RHS value for the other binding constraint in Flair's example—the carpentry constraint.

Increasing the RHS value of a nonbinding ≤ constraint does not affect the optimality of the current solution.

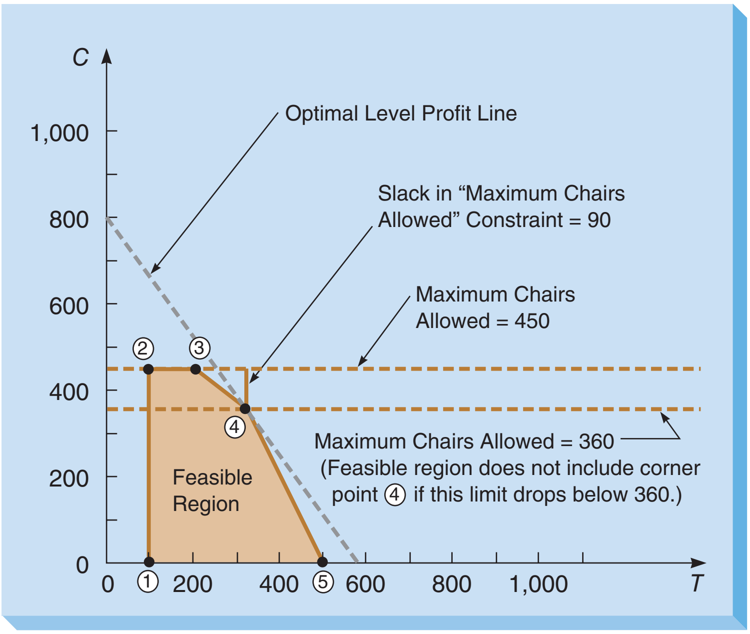
IMPACT OF CHANGES IN RHS VALUE OF A NONBINDING CONSTRAINT Let us now consider a nonbinding constraint such as the production limit on chairs ($C \leq 450$). As shown in Figure 4.8, the gap between corner point 4 and the chairs constraint represents the amount of **slack** in this nonbinding constraint. At the present solution, the slack is 90 ($= 450 - 360$). What happens now if the marketing department allows more chairs to be produced (i.e., the 450 limit is increased)? As we can see in Figure 4.8, such a change only serves to increase the slack in this constraint and does not affect the optimality of corner point 4 in any way. How far can we raise the 450 limit? Clearly, the answer is infinity.

The RHS value of a nonbinding ≤ constraint can be decreased up to its slack without affecting the optimality of the current solution.

Now consider the case where the marketing department wants to make this production limit even more restrictive (i.e., the 450 limit is decreased). As long as we are permitted to make at least 360 chairs, Figure 4.8 indicates that corner point 4 is feasible and still optimal. That is, as long as the change in the RHS value for the chairs constraint is within the slack of 90 units, the current optimal corner point continues to exist and remains optimal. However, if the chairs production limit is reduced below 360, corner point 4 is no longer feasible, and a new corner point becomes optimal. A similar analysis can be conducted with the other nonbinding constraint in the model (i.e., $T \geq 100$).

The preceding discussion illustrates that for nonbinding constraints, the allowable change limit on one side is infinity. On the opposite side, the allowable change limit equals the slack (or surplus).

FIGURE 4.8
Change in RHS Value of a Nonbinding Constraint



4.3 Sensitivity Analysis Using Solver Reports

Maximize \$7T + \$5C
subject to
$3T + 4C \leq 2{,}400$
$2T + 1C \leq 1{,}000$
$C \leq 450$
$T \geq 100$
$T, C \geq 0$

Let us consider Flair Furniture's LP model again (for your convenience, the formulation is shown in the margin note). Screenshot 4-1A shows the Excel layout and Solver entries for this model. Recall that we saw the same information in Chapter 2. Cells B5 and C5 are the entries in the By Changing Variable Cells box and denote the optimal quantities of tables and chairs to make, respectively. Cell D6 is the entry in the Set Objective box and denotes the profit. Cells D8 to D11 contain the formulas for the left-hand sides of each of the four constraints.

File: 4-1.xls, sheet: 4-1A

Excel Notes

- The Companion Website for this textbook, at www.pearsonhighered.com/balakrishnan, contains the Excel file for each problem in the examples discussed here. The relevant file name is shown in the margin next to each example.
- In each of our Excel layouts, for clarity, changing variable cells are shaded yellow, the objective cell is shaded green, and cells denoting left-hand-side (LHS) formulas of constraints are shaded blue. If the RHS of a constraint also includes a formula, that cell is also shaded blue.
- Also, to make the equivalence of the *written* formulation and the Excel layout clear, our Excel layouts show the decision variable names used in the written formulation of the model. Note that these names have no role in using Solver to solve the model.

Solver Reports

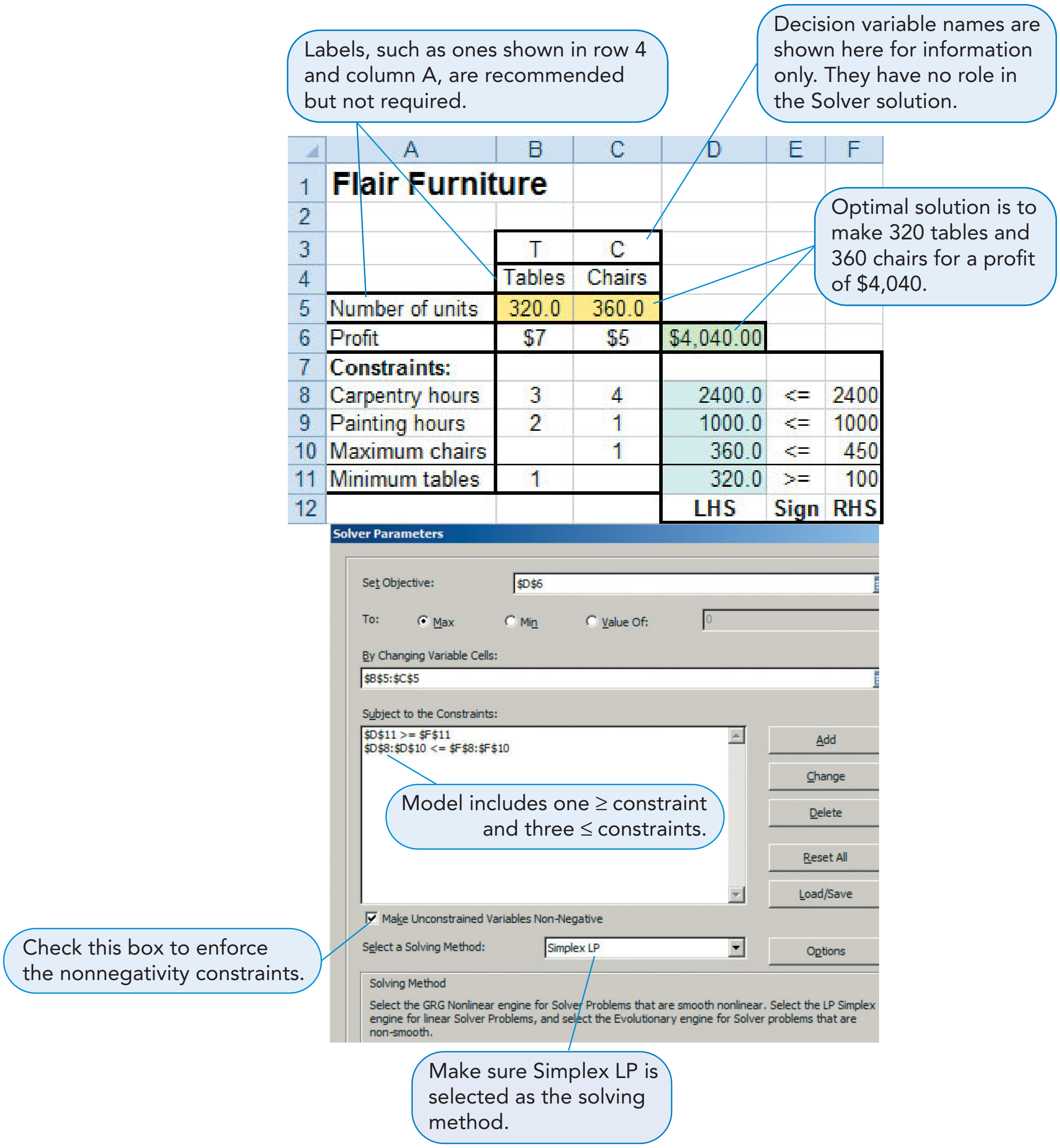
We must select the Simplex LP method as the solving method in the Solver Parameters window to obtain LP Sensitivity Reports.

Before solving the LP model, we need to ensure that the Simplex LP has been selected in the Select a Solving Method box in the Solver Parameters window to solve the problem (see Screenshot 4-1A). If a different method is selected, Solver does not solve the model as a linear program, and the resulting Sensitivity Report will look very different from the report we discuss here. Also, recall that we must check the Make Unconstrained Variables Non-Negative option to enforce the nonnegativity constraints.

The desired Solver reports must be selected in order for them to be created.

When Solver finds the optimal solution for a problem, the Solver Results window provides options to obtain three reports: Answer, Sensitivity, and Limits. Note that to obtain the desired reports, we must select them *before* we click OK. In our case, we select Answer and Sensitivity from the available choices in the box labeled Reports, and then click OK

SCREENSHOT 4-1A
Excel Layout and Solver Entries for Flair Furniture



(see Screenshot 4-1B). The Limits Report is relatively less useful, and we therefore do not discuss it here.

We discussed the **Answer Report** extensively in section 2.7 of Chapter 2 (see page 40) and urge you to read that section again at this time. Recall that this report provides essentially the same information as the original Excel layout (such as in Screenshot 4-1A) but in a more descriptive manner.

We are analyzing only one change at a time.

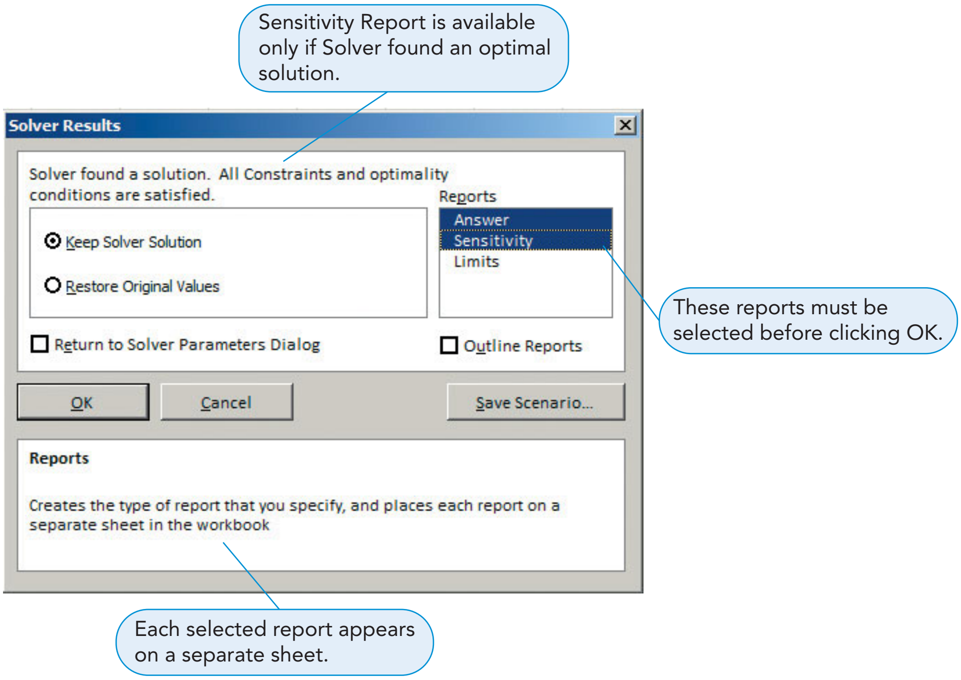
We now turn our attention to the information in the Sensitivity Report. Before we do so, it is important to note once again that while using the information in this report to answer what-if questions, we assume that we are considering a change to only a *single* input data value. Later, in section 4.5, we will expand our discussion to include simultaneous changes in several input data values.

Sensitivity Report

The Sensitivity Report for the Flair Furniture example is shown in Screenshot 4-1C. We have added grid lines to this report to make it clearer and have also formatted all values to display a consistent number of decimal points. The Sensitivity Report has two distinct tables, titled Variable Cells and Constraints. These tables permit us to answer several what-if questions regarding the problem solution.

File: 4-1.xls, sheet: 4-1C

SCREENSHOT 4-1B
Solver Results Window



Excel Note

Solver does a rather poor job of formatting the Sensitivity Report. There is no consistency in the number of decimal points shown. While some values are displayed with no decimal points, others are displayed with many decimal points. This could sometimes cause a value such as 0.35 to be displayed (and erroneously interpreted) as 0. For this reason, we urge you to format the Sensitivity Report as needed to display a consistent number of decimal points.

The Sensitivity Report has two parts: Variable Cells and Constraints.

The Variable Cells table presents information regarding the impact of changes to the OFCs (i.e., unit profits of $7 and $5) on the optimal solution. The Constraints table presents information related to the impact of changes in constraint RHS values (such as the 2,400 and 1,000 availabilities in carpentry and painting times, respectively) on the optimal solution. Although

SCREENSHOT 4-1C **Solver Sensitivity Report for Flair Furniture**

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [4-1.xls]Flair Furniture

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number of units Tables	320.00	0.00	7.00	3.00	3.25
$C$5	Number of units Chairs	360.00	0.00	5.00	4.33	1.50

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$8	Carpentry hours	2400.00	0.60	2400.00	225.00	900.00
$D$9	Painting hours	1000.00	2.60	1000.00	600.00	150.00
$D$10	Maximum chairs	360.00	0.00	450.00	1E+30	90.00
$D$11	Minimum tables	320.00	0.00	100.00	220.00	1E+30

Two components of the sensitivity report

The shadow prices are valid for this range of change in the RHS values.

Each additional hour of painting time will increase profit by $2.60.

The shadow price for a nonbinding constraint is zero.

Solver's way of showing infinity

different LP software packages may format and present these tables differently, the programs all provide essentially the same information.

Impact of Changes in a Constraint's RHS Value

Let us first discuss the impact on the optimal solution of a change in the RHS value of a constraint. As with the graph-based analysis earlier, we study this issue separately for binding and nonbinding constraints.

If the size of the feasible region increases, the optimal objective function value could improve.

IMPACT OF CHANGES IN THE RHS VALUE OF A BINDING CONSTRAINT Recall from the graph-based analysis in section 4.2 that if the RHS value of a binding constraint changes, the size of the feasible region also changes. If the change causes the feasible region to *increase* in size, the optimal objective function value could potentially improve. In contrast, if the change causes the feasible region to *decrease* in size, the optimal objective function value could potentially worsen. The magnitude of this change in the objective function value is given by the shadow price of the constraint, provided that the RHS change is within a certain range. In the Solver Sensitivity Report, this information is shown in the Constraints table in Screenshot 4-1C.

The shadow price is the change in objective function value for a one-unit increase in a constraint's RHS value.

Recall from section 4.2 that the shadow price can be defined as the change in the optimal objective function value for a one-unit increase in the RHS value of a constraint. In Screenshot 4-1C, the entry labeled Shadow Price for the painting constraint shows a value of \$2.60. This means that for each *additional* hour of painting time that Flair can obtain, its total profit changes by \$2.60. What is the direction of this change? In this specific case, the change is an increase in profit because the additional painting time causes the feasible region to become larger and, hence, the solution to improve.

VALIDITY RANGE FOR THE SHADOW PRICE For what level of increase in the RHS value of the painting constraint is the shadow price of \$2.60 valid? Once again, recall from our discussion in section 4.2 that there is a specific range of possible values for the RHS value of a binding constraint for which the current optimal corner point (i.e., the intersection point of the current binding constraints) exists, even if its actual location has changed. Increasing or decreasing the RHS value beyond this range causes this corner point to be no longer feasible and causes a new corner point to become the optimal solution.

The shadow price is valid only as long as the change in the RHS is within the Allowable Increase *and* Allowable Decrease *values.*

The information to compute the upper and lower limits of this range is given by the entries labeled Allowable Increase and Allowable Decrease in the Sensitivity Report. In Flair's case, these values show that the shadow price of \$2.60 for painting time availability is valid for an increase of up to 600 hours from the current value and a decrease of up to 150 hours. That is, the painting time available can range from a low of 850 (= 1,000 − 150) to a high of 1,600 (= 1,000 + 600) for the shadow price of \$2.60 to be valid. Note that the Allowable Decrease value implies that for each hour of painting time that Flair loses (up to 150 hours), its profit decreases by \$2.60. Likewise, the Allowable Increase value implies that for each hour of painting time that Flair gains (up to 600 hours), its profit increases by \$2.60.

The preceding discussion implies that if Flair can obtain an additional 300 hours of painting time, its profit will increase by 300 × \$2.60 = \$780, to \$4,820. In contrast, if it loses 100 hours of painting time, its profit will decrease by 100 × \$2.60 = \$260, to \$3,780. If the painting time availability increases by more than 600 hours (for example, increases by 700 hours, to 1,700 hours) or decreases by more than 150 hours (for example, decreases by 300 hours, to 700 hours) the current corner point is no longer feasible, and the solution has switched to a new corner point. Recall that we made these same observations earlier graphically using Figures 4.4 to 4.7.

For carpentry time, the shadow price is \$0.60, with a validity range of 1,500 (= 2,400 − 900) to 2,625 (= 2,400 + 225) hours. This means for every hour of carpentry time in this range, Flair's profit changes by \$0.60.

The shadow price of a nonbinding constraint is zero.

IMPACT OF CHANGES IN THE RHS VALUE OF A NONBINDING CONSTRAINT We note that Flair is planning to make only 360 chairs even though it is allowed to make as many as 450. Clearly, Flair's solution would not be affected in any way if we increased this production limit. Therefore, the shadow price for the chairs limit constraint is zero.

Solver displays infinity as 1E+30.

In Screenshot 4-1C, the allowable increase for this RHS value is shown to be infinity (displayed as 1E+30 in Solver). This is logical because any addition to the chair production limit will only cause the slack in this constraint to increase and will have no impact on profit. In contrast, once

SCREENSHOT 4-1D
Partial Solver Sensitivity Report for Flair Furniture

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [4-1.xls]Flair Furniture

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number of units Tables	320.00	0.00	7.00	3.00	3.25
$C$5	Number of units Chairs	360.00	0.00	5.00	4.33	1.50

Difference between marginal contribution and marginal worth of resources consumed.

Current OFC values

The current solution remains optimal for this range of change in OFC values.

we decrease this limit by 90 chairs (our current slack), this constraint also becomes binding. Any further reduction in this limit will clearly have an adverse effect on profit. This is revealed by the value of 90 for the allowable decrease in the RHS of the chairs limit constraint. To evaluate the new optimal solution if the production limit decreases by more than 90 chairs from its current value, the problem would have to be solved again.

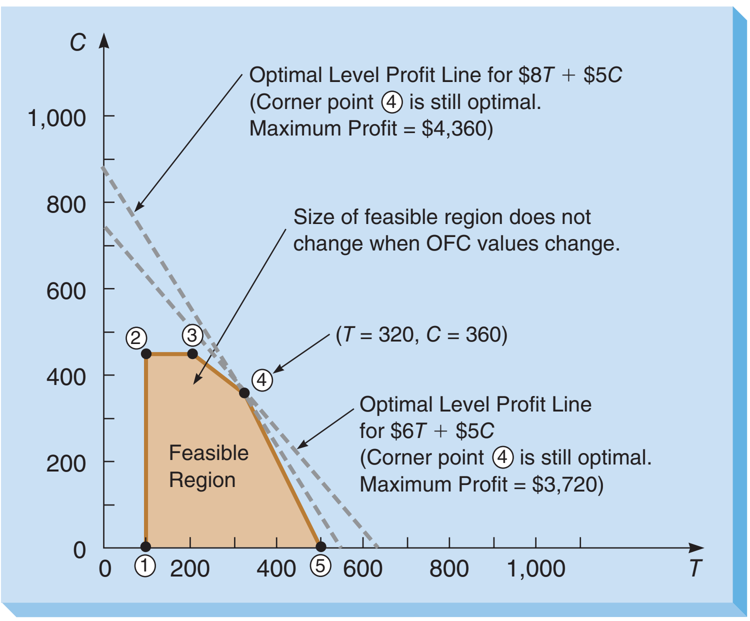
In a similar fashion, we note that Flair is planning to make 320 tables even though it is required to make only 100. Clearly, Flair's solution would not be affected in any way if we decreased this requirement from 100. This is indicated by the infinity in the Allowable Decrease column for this RHS. The current optimal solution will also not be affected as long as the increase in this RHS value is below 220. However, if Flair increases the RHS by more than 220 (and specifies that more than 320 tables must be made), the current optimal solution is no longer valid, and the model must be resolved to find the new solution.

Impact of Changes in an Objective Function Coefficient

Let us now focus on the information provided in the table titled Variable Cells. For your convenience, we repeat that part of Screenshot 4-1C here as Screenshot 4-1D. Each row in the Variable Cells table contains information regarding a decision variable in the model.

ALLOWABLE RANGES FOR OFCS In Figure 4.2, repeated here as Figure 4.9, we saw that as the unit profit contribution of either product changes, the slope of the isoprofit line changes. The size of the feasible region, however, remains the same. That is, the locations of the corner points do not change.

FIGURE 4.9
Changes in Profit Contribution of Tables



In the case of tables, as the unit profit increases from the current value of $7, the slope of the profit line in Figure 4.9 changes in a manner that makes corner point ④ an even more attractive optimal point. On the other hand, as the unit profit decreases, the slope of the profit line changes in a manner that makes corner point ③ become more and more attractive. At some point, the unit profit of tables is so low as to make corner point ③ the optimal solution.

There is an allowable decrease and an allowable increase for each OFC over which the current optimal solution remains optimal.

The limits to which the profit coefficient of tables can be changed without affecting the optimality of the current solution (corner point ④) is revealed by the values in the Allowable Increase and Allowable Decrease columns of the Sensitivity Report in Screenshot 4-1D. In the case of tables, their profit contribution per table can range anywhere from a low of $\$3.75\ (= \$7 - \$3.25)$ to a high of $\$10\ (= \$7 + \$3)$, and the current production plan $(T = 320, C = 360)$ will continue to remain optimal. The total profit will, of course, change, depending on the actual profit contribution per table. For example, if the profit contribution is $6 per table, the total profit is $\$3{,}720\ (= \$6 \times 320 + \$5 \times 360)$. This is the same result we saw earlier in Figure 4.2. Any profit contribution below $3.75 or over $10 per table will result in a different corner point solution being optimal.

For chairs, the profit contribution per chair can range anywhere from a low of $\$3.50\ (= \$5 - \$1.50)$ to a high of $\$9.33\ (= \$5 + \$4.33)$, and the current production plan will continue to remain optimal. Here again, the total profit will depend on the actual profit contribution per chair. For example, if the profit contribution is $8 per chair, the total profit is $\$5{,}120\ (= \$7 \times 320 + \$8 \times 360)$. Any profit contribution below $3.50 or over $9.33 per chair will result in a different corner point solution being optimal.

Reduced cost *is the difference between the marginal contribution of a variable and the marginal worth of the resources it uses.*

REDUCED COST The **Reduced Cost** values in Screenshot 4-1D show the difference between the marginal contribution of a decision variable to the objective function value (profit, in Flair's example) and the marginal worth of the resources it would consume if produced. A property of LP models is that if a variable has a nonzero value at optimality, its marginal contribution to the objective function value will equal the marginal worth of the resources it consumes. For instance, each table we produce uses 3 hours of carpentry time, uses 2 hours of painting time, counts 1 unit toward the 100-unit minimum tables requirement, and counts 0 units toward the 450-unit maximum chairs limit. Based on the preceding discussion of the shadow price, the marginal worth of these resources can be calculated as

$$\begin{aligned}
&= 3 \times \text{shadow price of carpentry constraint} + \\
&\quad\ 2 \times \text{shadow price of painting constraint} + \\
&\quad\ 1 \times \text{shadow price of minimum tables required constraint} + \\
&\quad\ 0 \times \text{shadow price of maximum chairs allowed constraint} \\
&= 3 \times \$0.6 + 2 \times \$2.6 + 1 \times \$0 + 0 \times \$0 \\
&= \$7
\end{aligned}$$

Note that this is equal to the profit contribution per table. The same calculation will hold for chairs also. The profit contribution per chair is $5, and the marginal worth of the resources it consumes is calculated as

$$\begin{aligned}
&= 4 \times \text{shadow price of carpentry constraint} + \\
&\quad\ 1 \times \text{shadow price of painting constraint} + \\
&\quad\ 0 \times \text{shadow price of minimum tables required constraint} + \\
&\quad\ 1 \times \text{shadow price of maximum chairs allowed constraint} \\
&= 4 \times \$0.6 + 1 \times \$2.6 + 0 \times \$0 + 1 \times \$0 \\
&= \$5
\end{aligned}$$

There is an alternate interpretation for the reduced cost that is relevant especially for decision variables with zero values in the current optimal solution. Because both variables in the Flair example had nonzero values at optimality, this interpretation was not relevant here. We will, however, see this alternate interpretation in the larger example we consider next.

4.4 Sensitivity Analysis for a Larger Maximization Example

Now that we have explained some of the basic concepts in sensitivity analysis, let us consider a larger production mix example that will allow us to discuss some further issues.

Anderson Home Electronics Example

This is a larger product mix example.

Anderson Home Electronics is considering the production of four inexpensive products for the low-end consumer market: an MP3 player, a satellite radio tuner, an LCD TV, and a Blu-Ray DVD player. For the sake of this example, let us assume that the input for all products can be viewed in terms of just three resources: electronic components, nonelectronic components, and assembly time. The composition of the four products in terms of these three inputs is shown in Table 4.1, along with the unit selling prices of the products.

Electronic components can be obtained at \$7 per unit; nonelectronic components can be obtained at \$5 per unit; assembly time costs \$10 per hour. Each resource is available in limited quantities during the upcoming production cycle, as shown in Table 4.1. Anderson believes the market demand is strong enough that it can sell all the quantities it makes of each product.

By subtracting the total cost of making a product from its unit selling price, the profit contribution of each product can be easily calculated. For example, the profit contribution of each MP3 player is \$29 (= selling price of \$70 less the total cost of $3 \times \$7 + 2 \times \$5 + 1 \times \$10$). Using similar calculations, see if you can confirm that the profit contribution of each satellite radio tuner is \$32, each LCD TV is \$72, and each Blu-Ray DVD player is \$54.

Let M, S, T, and B denote the number of MP3 players, satellite radio tuners, LCD TVs, and Blu-Ray DVD players to make, respectively. We can then formulate the LP model for this problem as follows:

$$\text{Maximize profit} = \$29M + \$32S + \$72T + \$54B$$

subject to the constraints

$$\begin{aligned} 3M + 4S + 4T + 3B &\le 4{,}700 && \text{(electronic components)} \\ 2M + 2S + 4T + 3B &\le 4{,}500 && \text{(nonelectronic components)} \\ M + S + 3T + 2B &\le 2{,}500 && \text{(assembly time, in hours)} \\ M, S, T, B &\ge 0 && \text{(nonnegativity)} \end{aligned}$$

File: 4-2.xls

Screenshots 4-2A, 4-2B, and 4-2C show the Excel layout and Solver entries, Answer Report, and Sensitivity Report, respectively, for Anderson's problem. The results show that Anderson should make 380 satellite radio tuners, 1,060 Blu-Ray DVD players, and no MP3 players or LCD TVs, for a total profit of \$69,400 in the upcoming production cycle.

Some Questions We Want Answered

We now ask and answer several questions that will allow us to understand the shadow prices, reduced costs, and allowable ranges information in the Anderson Home Electronics Sensitivity Report. Each question is independent of the other questions and assumes that only the change mentioned in that question is being considered.

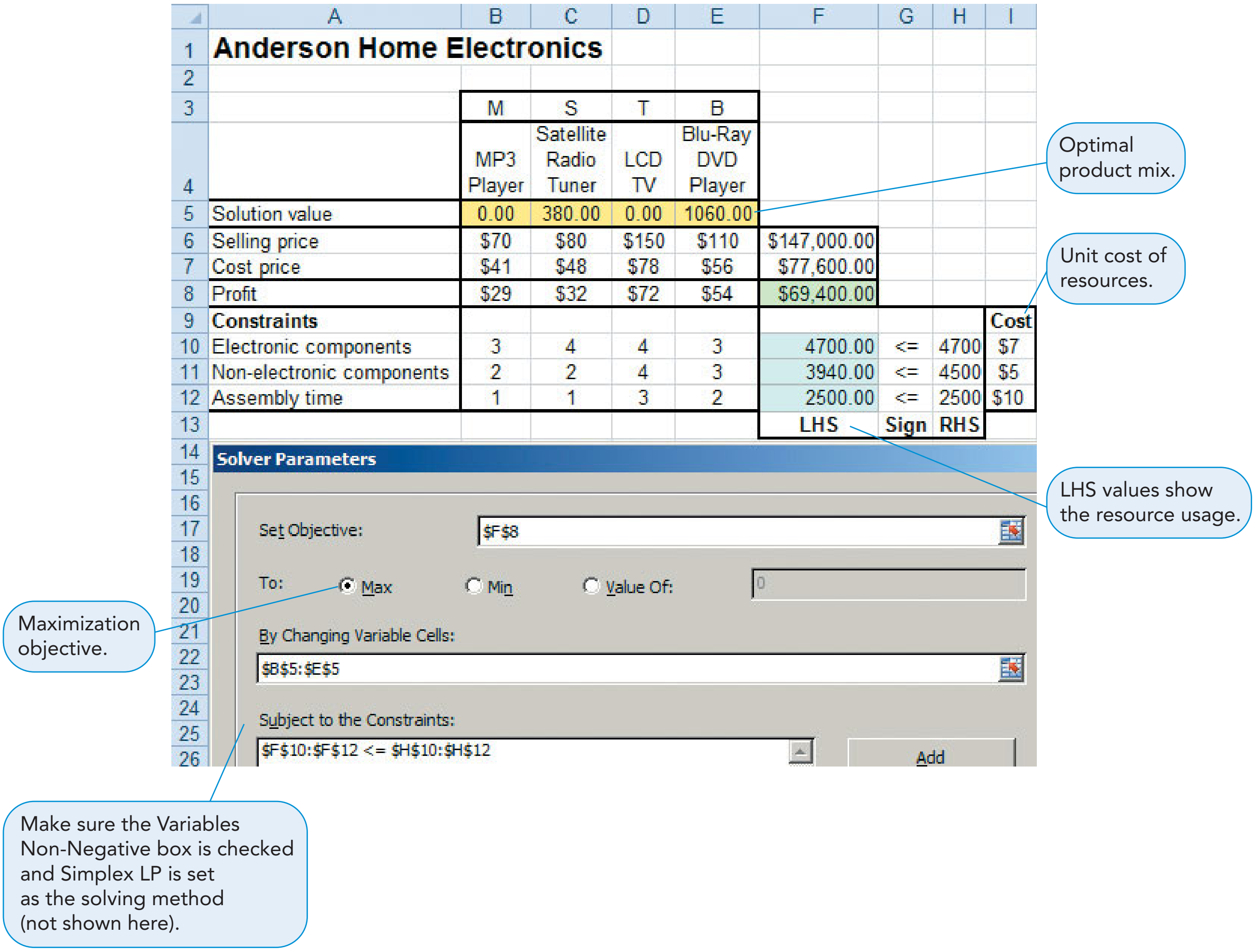
Nonelectronic components are a nonbinding constraint.

Q: What is the impact on profit of a change in the supply of nonelectronic components?
A: The slack values in the Answer Report (Screenshot 4-2B) indicate that of the potential supply of 4,500 units of nonelectronic components, only 3,940 units are used, leaving 560 units unused.

TABLE 4.1
Data for Anderson Home Electronics

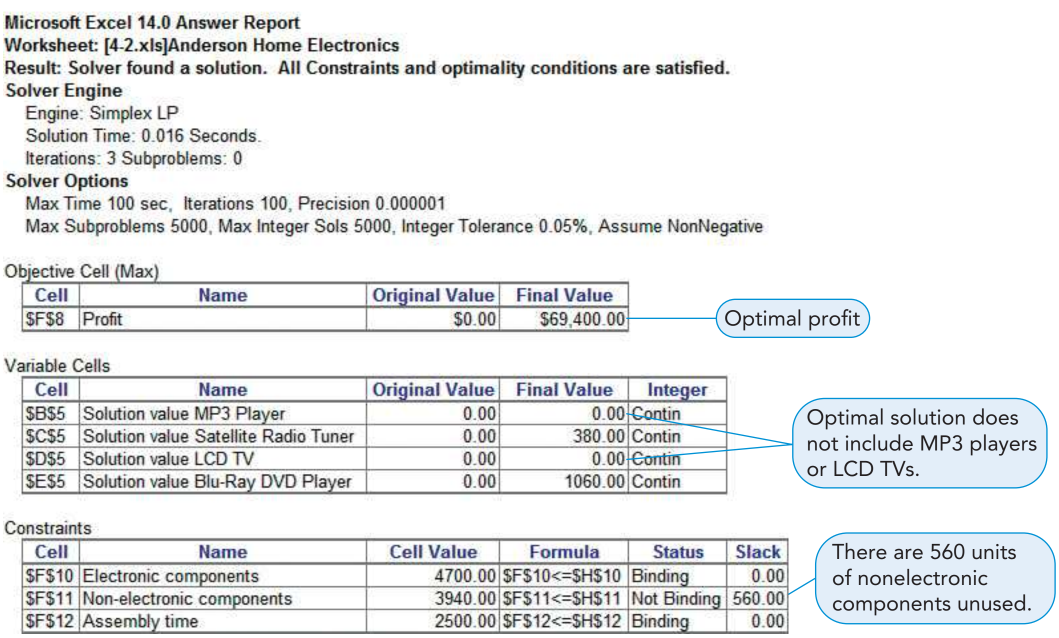
	MP3 PLAYER	SATELLITE RADIO TUNER	LCD TV	BLU-RAY DVD PLAYER	SUPPLY
Electronic components	3	4	4	3	4,700
Nonelectronic components	2	2	4	3	4,500
Assembly time (hours)	1	1	3	2	2,500
Selling price (per unit)	\$70	\$80	\$150	\$110	

SCREENSHOT 4-2A Excel Layout and Solver Entries for Anderson Home Electronics



	A	B	C	D	E	F	G	H	I
1	Anderson Home Electronics								
2									
3		M	S	T	B				
4		MP3 Player	Satellite Radio Tuner	LCD TV	Blu-Ray DVD Player				
5	Solution value	0.00	380.00	0.00	1060.00				
6	Selling price	$70	$80	$150	$110	$147,000.00			
7	Cost price	$41	$48	$78	$56	$77,600.00			
8	Profit	$29	$32	$72	$54	$69,400.00			
9	Constraints								Cost
10	Electronic components	3	4	4	3	4700.00	<=	4700	$7
11	Non-electronic components	2	2	4	3	3940.00	<=	4500	$5
12	Assembly time	1	1	3	2	2500.00	<=	2500	$10
13						LHS	Sign	RHS	

SCREENSHOT 4-2B Solver Answer Report for Anderson Home Electronics



Microsoft Excel 14.0 Answer Report
Worksheet: [4-2.xls]Anderson Home Electronics
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.016 Seconds.
Iterations: 3 Subproblems: 0
Solver Options
Max Time 100 sec, Iterations 100, Precision 0.000001
Max Subproblems 5000, Max Integer Sols 5000, Integer Tolerance 0.05%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$F$8	Profit	$0.00	$69,400.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$5	Solution value MP3 Player	0.00	0.00	Contin
$C$5	Solution value Satellite Radio Tuner	0.00	380.00	Contin
$D$5	Solution value LCD TV	0.00	0.00	Contin
$E$5	Solution value Blu-Ray DVD Player	0.00	1060.00	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$10	Electronic components	4700.00	$F$10<=$H$10	Binding	0.00
$F$11	Non-electronic components	3940.00	$F$11<=$H$11	Not Binding	560.00
$F$12	Assembly time	2500.00	$F$12<=$H$12	Binding	0.00

SCREENSHOT 4-2C **Solver Sensitivity Report for Anderson Home Electronics**

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [4-2.xls]Anderson Home Electronics

For changes to OFC values in this range, current solution remains optimal.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Solution value MP3 Player	0.00	-1.00	29.00	1.00	1E+30
$C$5	Solution value Satellite Radio Tuner	380.00	0.00	32.00	40.00	1.67
$D$5	Solution value LCD TV	0.00	-8.00	72.00	8.00	1E+30
$E$5	Solution value Blu-Ray DVD Player	1060.00	0.00	54.00	10.00	5.00

Allowable decrease is infinity since product is not attractive even at current OFC value.

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$10	Electronic components	4700.00	2.00	4700.00	2800.00	950.00
$F$11	Non-electronic components	3940.00	0.00	4500.00	1E+30	560.00
$F$12	Assembly time	2500.00	24.00	2500.00	466.67	1325.00

Since nonelectronic components are nonbinding, shadow price is zero.

The allowable increase is infinity since there are already 560 units of slack.

This implies that additional nonelectronic components are of no value to Anderson in terms of contribution to profit; that is, the shadow price is zero.

This shadow price is valid for an unlimited (infinite) increase in the supply of nonelectronic components. Further, Anderson would be willing to give up as many as 560 units of these components with no impact on profit. These values are shown in the Allowable Increase and Allowable Decrease columns in Screenshot 4-2C, respectively, for the supply of nonelectronic components.

Q: What is the impact on profit if we could increase the supply of electronic components by 400 units (to a total of 5,100 units)?

Electronic components are a binding constraint.

A: We first look at the Allowable Increase column for electronic components in Screenshot 4-2C to verify whether the current shadow price is valid for an increase of 400 units in this resource. Because the Allowable Increase column shows a value of 2,800, the shadow price is valid.

Next, we look at the shadow price for electronic components, which is $2 per unit. That is, each additional unit of electronic components (up to 2,800 additional units) will allow Anderson to increase its profit by $2. The impact of 400 units will therefore be a net increase in profit of $800. The new profit will be $70,200 (= $69,400 + $800).

A change in the RHS value of a binding constraint causes the coordinates of the optimal corner point to change.

It is important to remember that whenever the RHS value of a nonredundant constraint changes, the size of the feasible region changes. Hence, some of the corner points shift locations. In the current situation, because the proposed change is within the allowable change, the current corner point is still *optimal*. That is, the constraints that are binding at present will continue to remain the binding constraints. However, the corner point itself has shifted from its present location. What are the values of the decision variables at the new location of this corner point? Because we know which constraints are binding at the optimal point, we can answer this question by solving those equations simultaneously. Alternatively, we can resolve the LP model.

Q: In the previous question, what would happen if we could increase the supply of electronic components by 4,000 units (to a total of 8,700 units)?

Changes beyond the allowable increase or decrease cannot be analyzed using the current report.

A: From Screenshot 4-2C, we see that the shadow price of $2 per unit is valid only up to 2,800 additional units. This means that the first 2,800 units will cause the total profit to increase by $5,600 (= $2 × 2,800). However, the impact of the last 1,200 units (assuming that we are forced to accept all or nothing of the 4,000 units) cannot be analyzed by using the current report. The problem would have to be resolved using Solver to measure its impact.

The fact that the potential additional supply (4,000) of electronic components is beyond the allowable increase value (2,800) does *not* mean that Anderson's management cannot implement this change. It just means that the total impact of the change cannot be evaluated from the *current* Sensitivity Report in Screenshot 4-2C.

Q: Refer to the question about getting an additional 400 units of electronic components. What would happen if the supplier of these 400 units wanted $8 per unit rather than the current cost of $7 per unit?

We must correct the shadow price for any premium that we pay.

A: We know that the shadow price of $2 for electronic components represents the increase in total profit from each additional unit of this resource. This value is net after the cost of this additional unit has been taken into account. That is, it is actually beneficial for Anderson to pay a premium of up to $2 per additional unit of electronic components. In the current situation, getting 400 additional units of electronic components would cost Anderson $8 per unit. This represents a premium of $1 per unit over the current rate of $7 per unit. However, it would still be beneficial to get these units because each additional unit would increase the total profit by $1 (= shadow price of $2 less the premium of $1). The total profit would therefore increase by $400, to a new value of $69,800.

This adjusted value of $1 represents the actual increase in profit and can be referred to as the *adjusted shadow price*.

Q: Assume that we have an opportunity to get 250 additional hours of assembly time. However, this time will cost us time and a half (i.e., $15 per hour rather than the current $10 per hour). Should we take it?

We must calculate the adjusted shadow price here.

A: From Screenshot 4-2C, the shadow price of $24 per hour of assembly time is valid for an increase of up to 466.67 hours. This shadow price, however, assumes that the additional time costs only $10 per hour. The $5 per hour premium paid on the additional time therefore results in an increase of only $19 (= $24 − $5) per each additional hour of assembly time obtained.

The net impact on profit of the additional 250 hours of assembly time is an increase of $4,750 (= 250 × $19). Anderson should definitely accept this opportunity.

Q: If we force the production of MP3 players, what would be the impact on total profit?

The impact of forcing MP3 players to be produced is shown by the reduced cost.

A: MP3 players are currently not being recommended for production because they are not profitable enough. You may recall from our discussion in section 4.3 that the reduced cost shows the difference between the marginal contribution of a product to the objective function (profit contribution is $29 per MP3 player, in Anderson's case) and the marginal worth of resources it would consume if produced. As an exercise, see if you can verify that this value is $30 per MP3 player. The reduced cost for MP3 player is therefore −$1 (= $29 − $30), as shown in Screenshot 4-2C. This implies that the net impact of producing one MP3 player will be to decrease total profit by $1 (to $69,399).

Q: How profitable must MP3 players become before Anderson would consider producing them?

A: We know that each MP3 player produced will cause Anderson's profit to decrease by $1. This implies that if Anderson can find a way of increasing the profit contribution of MP3 players by $1, MP3 players would then become an attractive product. This can be achieved either by increasing the selling price of MP3 players by $1 (to $71 per unit) or by reducing their cost price by $1, or a combination of the two.

Reduced cost is also the minimum amount by which the OFC of a variable should change in order to affect the optimal solution.

This is an alternate interpretation of reduced cost. That is, the magnitude of the reduced cost is the minimum amount by which the OFC of a variable should change in order for it to affect the optimal solution. For MP3 players, if their OFC increases by more than $1 per unit, MP3 players will then have a nonzero value in the new optimal solution.

This information is also seen from the $1 in the Allowable Increase column for the OFC for MP3 players. Not surprisingly, the Allowable Decrease column shows a value of infinity (shown as 1E+30 in Excel) for the OFC of MP3 players. This is logical because if MP3 players are not attractive at a unit profit of $29, they are clearly not going to be attractive at unit profit values lower than $29.

Q: Assume that there is some uncertainty in the price for Blu-Ray DVD players. For what range of prices will the current production be optimal? If Blu-Ray DVD players sold for $106, what would be Anderson's new total profit?

Even though the production values do not change, the total profit will decrease.

A: Blu-Ray DVD players currently sell for \$110, yielding a profit of \$54 per unit. The allowable ranges for the OFC of Blu-Ray DVD players in Screenshot 4-2C shows that this value can increase by up to \$10 (to \$64; selling price of \$120) or decrease by up to \$5 (to \$49; selling price of \$105) for the current production plan to remain optimal.

If Blu-Ray DVD players actually sold for \$106, the profit per unit would drop to \$50. The current values of the decision variables would remain optimal. However, the new total profit would decrease by \$4,240 (= \$4 per Blu-Ray DVD player for 1,060 players), to \$65,160.

Alternate Optimal Solutions

Is the optimal solution identified in Screenshot 4-2A for Anderson Home Electronics (380 satellite radio tuners and 1,060 Blu-Ray DVD players, for a total profit of \$69,400) unique? Are there alternate production mixes that will also yield a profit of \$69,400?

Recall that in Chapter 2 (section 2.6, on page 38) we saw a graphical example of a situation in which a problem with only two variables had alternate optimal solutions (also referred to as *multiple optimal solutions*). How can we detect a similar condition from the Solver Sensitivity Report for problems involving more than two variables?

Zeros in the Allowable Increase or Allowable Decrease columns for OFC values may indicate alternate optimal solutions.

In most cases, when the Allowable Increase or Allowable Decrease value for the OFC of a variable is zero in the Variable Cells table, this indicates the presence of alternate optimal solutions. In Anderson's problem, we see from Screenshot 4-2C that this is not the case.

Note also from Screenshot 4-2C that the reduced costs for both products currently not being produced in the optimal solution (MP3 players and LCD TVs) are nonzero. This indicates that if Anderson is forced to produce either of these products, the net impact will be a reduction in total profit (as discussed earlier). That is, there is no solution possible involving products other than satellite radio tuners and Blu-Ray DVD players that will yield a profit as high as the current solution (\$69,400). The current optimal solution is, therefore, unique.

In Solved Problem 4-1 at the end of this chapter, we discuss a problem for which the Solver Sensitivity Report indicates the presence of alternate optimal solutions. We also discuss how Solver can be used to identify these alternate optimal solutions.

4.5 Analyzing Simultaneous Changes by Using the 100% Rule

Until now, we have analyzed the impact of a change in just a single parameter value on the optimal solution. That is, when we are studying the impact of one item of the input data (OFC or RHS value), we assume that all other input data in the model stay constant at their current values. What happens when there are *simultaneous* changes in more than one OFC value or more than one RHS value? Is it possible to analyze the impact of such simultaneous changes on the optimal solution with the information provided in the Sensitivity Report?

The answer is yes, albeit under only a specific condition, as discussed in the following section. It is important to note that the condition is valid only for analyzing simultaneous changes in either OFC values or RHS values, but not a mixture of the two types of input data.

Simultaneous Changes in Constraint RHS Values

Consider a situation in which Anderson Home Electronics realizes that its available number of electronic components is actually only 4,200 and, *at the same time*, also finds that it has an opportunity to obtain an additional 200 hours of assembly time. What is the impact of these *simultaneous* changes on the optimal solution? To answer this question, we first use a condition called the **100% rule**. This condition can be stated as follows:

$$\sum_{\text{changes}} (\text{change}/\text{allowable change}) \le 1 \tag{4-1}$$

The 100% rule can be used to check whether simultaneous changes in RHS or OFC values can be analyzed by using the current Sensitivity Report.

That is, we compute the ratio of each proposed change in a parameter's value to the maximum allowable change in its value, as given in the Sensitivity Report. The sum of these ratios must not exceed 1 (or 100%) in order for the information given in the current Sensitivity Report to be valid. If the sum of the ratios does exceed 1, the current information may still be valid; we just cannot guarantee its validity. However, if the ratio does not exceed 1, the information is definitely valid.

To verify this rule for the proposed change in Anderson's problem, consider each change in turn. First, there is a decrease of 500 units (i.e., from 4,700 to 4,200) in the number of electronic components. From the Sensitivity Report (see Screenshot 4-2C), we see that the allowable decrease in this RHS value is 950. The ratio is therefore

$$500/950 = 0.5263$$

Next, there is an increase of 200 hours (from 2,500 to 2,700) in the assembly time available. From the Sensitivity Report, we see that the allowable increase for this RHS value is 466.67. This ratio is, therefore,

$$200/466.67 = 0.4285$$

The sum of these ratios is

$$\text{Sum of ratios} = 0.5263 + 0.4285 = 0.9548 < 1$$

If the sum of ratios does not exceed 1, the information in the Sensitivity Report is valid.

Because this sum does not exceed 1, the information provided in the Sensitivity Report is valid for analyzing the impact of these changes. First, the decrease of 500 units in electronic component availability reduces the size of the feasible region and will therefore cause profit to decrease. The magnitude of this decrease is \$1,000 (= 500 units of electronic components, at a shadow price of \$2 per unit).

In contrast, the additional 200 hours of assembly time will result in a larger feasible region and a net increase in profit of \$4,800 (= 200 hours of assembly time, at a shadow price of \$24 per hour). The net impact of these simultaneous changes is therefore an increase in profit of \$3,800 (= \$4,800 − \$1,000).

Simultaneous Changes in OFC Values

The 100% rule can also be used to analyze simultaneous changes in OFC values in a similar manner. For example, what is the impact on the optimal solution if Anderson decides to drop the selling price of Blu-Ray DVD players by \$3 per unit but, at the same time, increase the selling price of satellite radio tuners by \$8 per unit?

Once again, we calculate the appropriate ratios to verify the 100% rule. For the current solution to remain optimal, the allowable decrease in the OFC for Blu-Ray DVD players is \$5, while the allowable increase in the OFC for satellite radio tuners is \$40. The sum of ratios is therefore

$$\text{Sum of ratios} = (\$3/\$5) + (\$8/\$40) = 0.80 < 1$$

Because the sum of ratios does not exceed 1, the current production plan is still optimal. The \$3 decrease in profit per Blu-Ray DVD player causes total profit to decrease by \$3,180 (= \$3 × 1,060). However, the \$8 increase in the unit profit of each satellite radio tuner results in an increase of \$3,040 (= \$8 × 380) in total profit. The net impact is, therefore, a decrease in profit of only \$140, to a new value of \$69,260.

4.6 Pricing Out New Variables

Pricing out analyzes the impact of adding a new variable to the existing LP model.

The information given in the Sensitivity Report can also be used to study the impact of the introduction of new decision variables (products, in the Anderson example) in the model. For example, if Anderson's problem is solved again with a new product also included in the model, will we recommend that the new product be made? Or will we recommend not to make the new product and continue to make the same products (i.e., satellite radio tuners and Blu-Ray DVD players) Anderson is making now?

Anderson's Proposed New Product

Suppose Anderson Home Electronics wants to introduce a new product, the Digital Home Theater Speaker System (DHTSS), to take advantage of the hot market for that product. The design department estimates that each DHTSS will require five units of electronic components, four units of nonelectronic components, and four hours of assembly time. The marketing department estimates that it can sell each DHTSS for \$175, a slightly higher selling price than any of the other four products being considered by Anderson.

The question now is whether the DHTSS will be a profitable product for Anderson to produce. That is, even though the new product would have a higher selling price per unit, is it worthwhile from an overall profit perspective to divert resources from Anderson's existing products to make this new product? Alternatively, we could pose the question as this: What is the minimum price at which Anderson would need to sell each DHTSS in order to make it a viable product?

The answer to such a question involves a procedure called **pricing out**. Assume that Anderson decides to make a single DHTSS. Note that the resources required to make this system (five units of electronic components, four units of nonelectronic components, and four hours of assembly time) will no longer be available to meet Anderson's existing production plan (380 satellite radio tuners and 1,060 Blu-Ray DVD players, for a total profit of $69,400).

CHECKING THE VALIDITY OF THE 100% RULE Clearly, the loss of these resources is going to reduce the profit that Anderson could have made from its existing products. Using the shadow prices of these resources, we can calculate the exact impact of the loss of these resources. However, we must first use the 100% rule to check whether the shadow prices are valid by calculating the ratio of the reduction in each resource's availability to the allowable decrease for that resource (given in Screenshot 4-2C on page 136). The resulting calculation is as follows:

$$\text{Sum of ratios} = (5/950) + (4/560) + (4/1{,}325) = 0.015 < 1$$

REQUIRED PROFIT CONTRIBUTION OF EACH DHTSS Because the total ratio is less than 1, the shadow prices are valid to calculate the impact on profit of using these resources to produce a DHTSS, rather than the existing products. We can determine this impact as

We first calculate the worth of the resources that would be consumed by the new product, if produced.

$$\begin{aligned} &= 5 \times \text{shadow price of electronic components constraint} + \\ &\quad 4 \times \text{shadow price of nonelectronic components constraint} + \\ &\quad 4 \times \text{shadow price of assembly time constraint} \\ &= 5 \times \$2 + 4 \times \$0 + 4 \times \$24 \\ &= \$106 \end{aligned}$$

Hence, in order for DHTSS to be a viable product, the profit contribution of each DHTSS has to at least make up this shortfall in profit. That is, the OFC for DHTSS must be at least $106 in order for the optimal solution to have a nonzero value for DHTSS. Otherwise, the optimal solution of Anderson's model with a decision variable for DHTSS included will be the same as the current one, with DHTSS having a value of zero.

FINDING THE MINIMUM SELLING PRICE OF EACH DHTSS UNIT The actual cost of the resources used to make one DHTSS unit can be calculated as

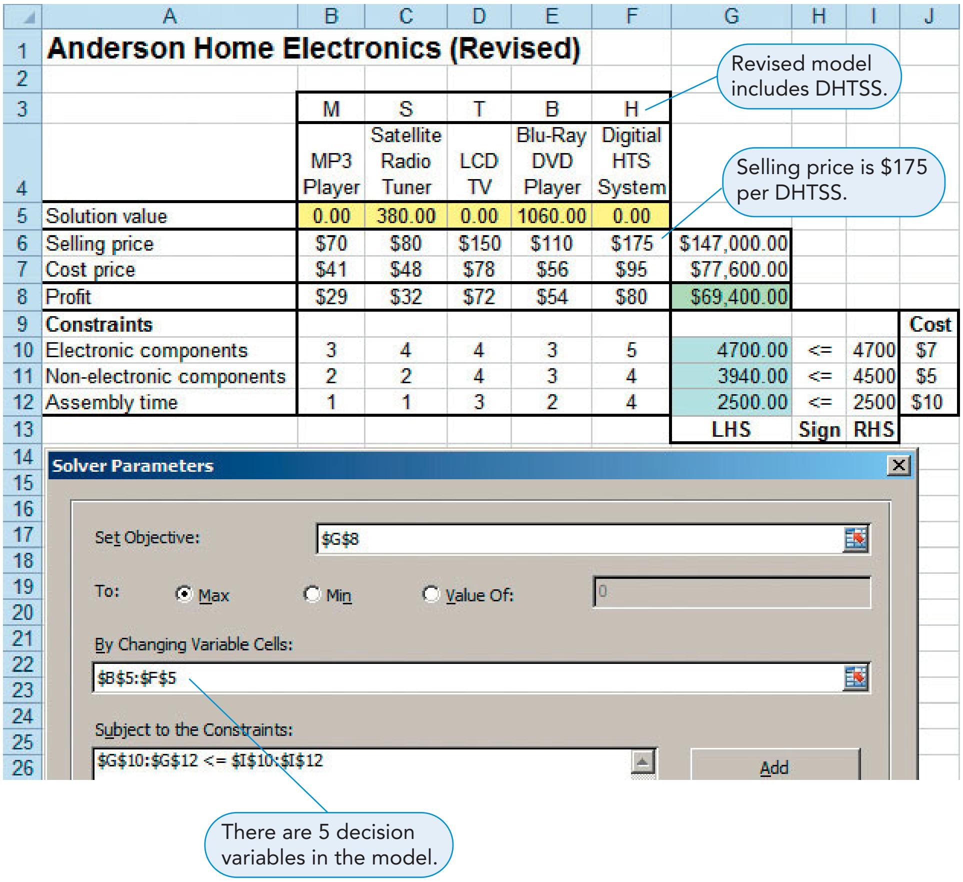
We calculate the actual cost of making the new product, if produced.

$$\begin{aligned} &= 5 \times \text{unit price of electronic components constraint} + \\ &\quad 4 \times \text{unit price of nonelectronic components constraint} + \\ &\quad 4 \times \text{unit price of assembly time constraint} \\ &= 5 \times \$7 + 4 \times \$5 + 4 \times \$10 \\ &= \$95 \end{aligned}$$

The minimum selling price for DHTSS units is then calculated as the sum of the cost of making a DHTSS unit and the marginal worth of resources diverted from existing products. In Anderson's case, this works out to $201 (= $106 + $95). Because Anderson's marketing department estimates that it can sell each DHTSS unit for only $175, this product will not be profitable for Anderson to produce.

What happens if Anderson *does* include DHTSS as a variable in its model and solves the expanded formulation again? In this case, from the discussion so far, we can say that the optimal solution will once again recommend producing 380 satellite radio tuners and 1,060 Blu-Ray DVD players, for a total profit of $69,400. DHTSS will have a final value of zero (just as MP3 players and LCD TVs do in the current solution). What will be the reduced cost of DHTSS in this revised solution? We have calculated that the minimum selling price required for DHTSS to be a viable product is $201, while the actual selling price is only $175. Therefore, the reduced cost will be −$26, indicating that each DHTSS unit produced will cause Anderson's profit to decrease by $26.

SCREENSHOT 4-3A
Excel Layout and Solver Entries for Anderson Home Electronics—Revised Model



	A	B	C	D	E	F	G	H	I	J
1	Anderson Home Electronics (Revised)									
2										
3		M	S	T	B	H				
4		MP3 Player	Satellite Radio Tuner	LCD TV	Blu-Ray DVD Player	Digitial HTS System				
5	Solution value	0.00	380.00	0.00	1060.00	0.00				
6	Selling price	$70	$80	$150	$110	$175	$147,000.00			
7	Cost price	$41	$48	$78	$56	$95	$77,600.00			
8	Profit	$29	$32	$72	$54	$80	$69,400.00			
9	**Constraints**									**Cost**
10	Electronic components	3	4	4	3	5	4700.00	<=	4700	$7
11	Non-electronic components	2	2	4	3	4	3940.00	<=	4500	$5
12	Assembly time	1	1	3	2	4	2500.00	<=	2500	$10
13							LHS	Sign	RHS	

File: 4-3.xls

To verify our conclusions, let us revise the LP model for Anderson Home Electronics to include the new product, DHTSS. The Excel layout and Solver entries for this revised model are shown in Screenshot 4-3A. The Sensitivity Report for this model is shown in Screenshot 4-3B.

The results show that it continues to be optimal for Anderson to produce 380 satellite radio tuners and 1,060 Blu-Ray DVD players, for a total profit of $69,400. Further, the magnitude of the reduced cost for HTS is $26, as we had already calculated.

SCREENSHOT 4-3B **Solver Sensitivity Report for Anderson Home Electronics—Revised Model**

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [4-3.xls]Anderson Home Electronics (Revised)

This is the same product mix as in Screenshot 4-2C.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Solution value MP3 Player	0.00	-1.00	29.00	1.00	1E+30
$C$5	Solution value Satellite Radio Tuner	380.00	0.00	32.00	40.00	1.67
$D$5	Solution value LCD TV	0.00	-8.00	72.00	8.00	1E+30
$E$5	Solution value Blu-Ray DVD Player	1060.00	0.00	54.00	10.00	5.00
$F$5	Solution value Digitial HTS System	0.00	-26.00	80.00	26.00	1E+30

DHTSS is not included in the optimal product mix.

Reduced cost for DHTSS is −$26.

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$G$10	Electronic components	4700.00	2.00	4700.00	2800.00	950.00
$G$11	Non-electronic components	3940.00	0.00	4500.00	1E+30	560.00
$G$12	Assembly time	2500.00	24.00	2500.00	466.67	1325.00

These are the same shadow prices as in Screenshot 4-2C.

4.7 Sensitivity Analysis for a Minimization Example

Minimization problems typically involve some ≥ constraints.

Let us now analyze an example with a minimization objective. For such problems, we need to be aware that when a solution *improves*, the objective value decreases rather than increases.

Burn-Off Diet Drink Example

Burn-Off, a manufacturer of diet drinks, is planning to introduce a miracle drink that will magically burn away fat. The drink is a bit expensive, but Burn-Off guarantees that a person using this diet plan will lose up to 50 pounds in just three weeks. The drink is made up of four "mystery" ingredients (which we will call ingredients A, B, C, and D). The plan calls for a person to consume at least three 12-ounce doses per day (i.e., at least 36 ounces per day) but no more than 40 ounces per day.

Each of the four ingredients contains different levels of three chemical compounds (which we will call chemicals X, Y, and Z). Health regulations mandate that the dosage consumed per day should contain minimum prescribed levels of chemicals X and Y and should not exceed maximum prescribed levels for the third chemical, Z.

The composition of the four ingredients in terms of the chemical compounds (units per ounce) is shown in Table 4.2, along with the unit cost prices of the ingredients. Burn-Off wants to find the optimal way to mix the ingredients to create the drink, at minimum cost per daily dose.

To formulate this problem, we let A, B, C, and D denote the number of ounces of ingredients A, B, C, and D to use, respectively. The problem can then be formulated as follows:

$$\text{Minimize daily dose cost} = \$0.40A + \$0.20B + \$0.60C + \$0.30D$$

subject to the constraints

$$\begin{aligned}
A + B + C + D &\geq 36 && \text{(daily dosage minimum)}\\
3A + 4B + 8C + 10D &\geq 280 && \text{(chemical X requirement)}\\
5A + 3B + 6C + 6D &\geq 200 && \text{(chemical Y requirement)}\\
10A + 25B + 20C + 40D &\leq 1{,}050 && \text{(chemical Z max limit)}\\
A + B + C + D &\leq 40 && \text{(daily dosage maximum)}\\
A, B, C, D &\geq 0 && \text{(nonnegativity)}
\end{aligned}$$

Burn-Off's Excel Solution

File: 4-4.xls

Screenshots 4-4A and 4-4B show the Excel layout and Solver entries, and the Sensitivity Report, respectively, for Burn-Off's problem. The output shows that the optimal solution is to use 10.25 ounces of ingredient A, 4.125 ounces of ingredient C, and 21.625 ounces of ingredient D, to make exactly 36 ounces of the diet drink per day. Interestingly, ingredient B is not used, even though it is the least expensive ingredient. The total cost is $13.06 per day.

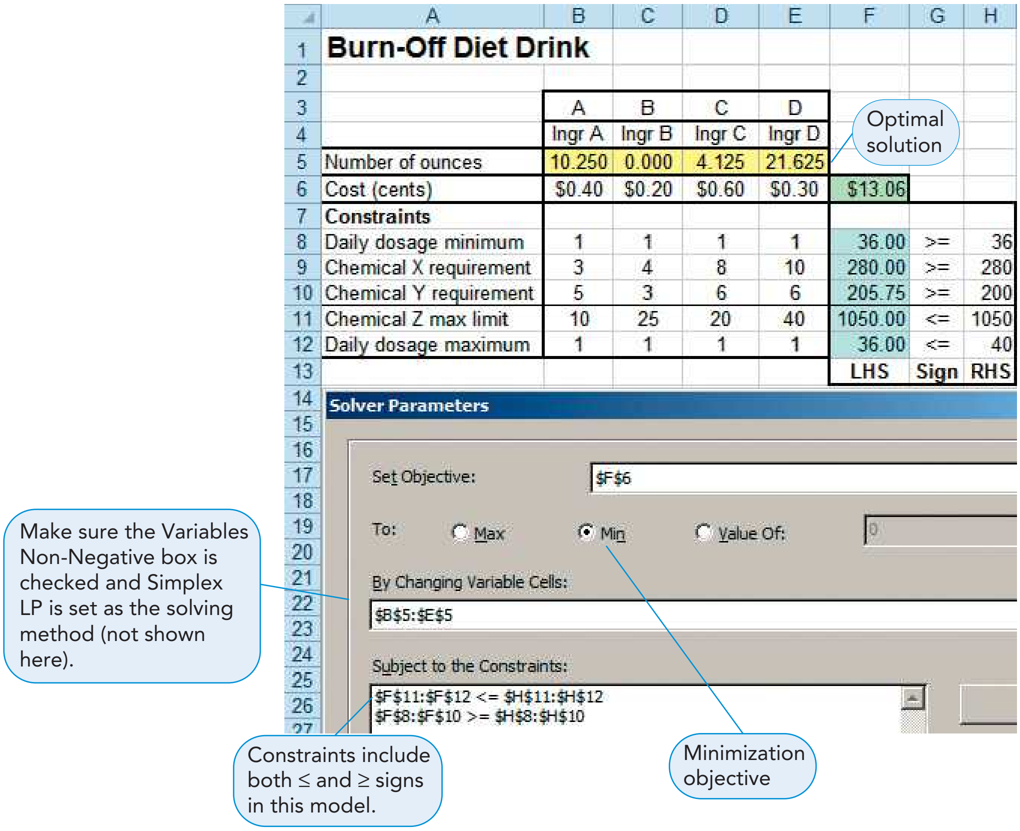
The difference between the LHS and RHS values of a ≥ constraint is called **surplus.**

The solution also indicates that the constraints for chemical Y and the maximum daily dosage are nonbinding. Although the minimum requirement is for only 200 units of chemical Y, the final drink actually provides 205.75 units of this chemical. The extra 5.75 units denote the level of oversatisfaction of this requirement. You may recall from Chapter 2 that we refer to this quantity as **surplus**, even though the Solver Answer Report always titles this value *slack*.

TABLE 4.2 Data for Burn-Off Diet Drink

	INGREDIENT A	INGREDIENT B	INGREDIENT C	INGREDIENT D	REQUIREMENT
Chemical X	3	4	8	10	At least 280 units
Chemical Y	5	3	6	6	At least 200 units
Chemical Z	10	25	20	40	At most 1,050 units
Cost per ounce	$0.40	$0.20	$0.60	$0.30	

SCREENSHOT 4-4A
Excel Layout and Solver Entries for Burn-Off Diet Drink



	A	B	C	D	E	F	G	H
1	Burn-Off Diet Drink							
2								
3		A	B	C	D			
4		Ingr A	Ingr B	Ingr C	Ingr D			
5	Number of ounces	10.250	0.000	4.125	21.625			
6	Cost (cents)	$0.40	$0.20	$0.60	$0.30	$13.06		
7	Constraints							
8	Daily dosage minimum	1	1	1	1	36.00	>=	36
9	Chemical X requirement	3	4	8	10	280.00	>=	280
10	Chemical Y requirement	5	3	6	6	205.75	>=	200
11	Chemical Z max limit	10	25	20	40	1050.00	<=	1050
12	Daily dosage maximum	1	1	1	1	36.00	<=	40
13						LHS	Sign	RHS

Answering Sensitivity Analysis Questions for Burn-Off

As with the Anderson Home Electronics example, we use several questions to interpret the information given in the Sensitivity Report (Screenshot 4-4B) for Burn-Off.

Q: What is the impact on cost if Burn-Off insists on using 1 ounce of ingredient B to make the drink?

SCREENSHOT 4-4B
Solver Sensitivity Report for Burn-Off Diet Drink

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [4-4.xls]Burn-Off Diet Drink

Reduced cost shows increase in total cost if ingredient B is used.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number of ounces Ingr A	10.250	0.000	0.400	0.061	0.250
$C$5	Number of ounces Ingr B	0.000	0.069	0.200	1E+30	0.069
$D$5	Number of ounces Ingr C	4.125	0.000	0.600	1.500	0.073
$E$5	Number of ounces Ingr D	21.625	0.000	0.300	0.085	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$8	Daily dosage minimum	36.000	0.375	36.00	16.500	1.278
$F$9	Chemical X requirement	280.000	0.088	280.000	41.000	11.000
$F$10	Chemical Y requirement	205.750	0.000	200.000	5.750	1E+30
$F$11	Chemical Z max limit	1050.000	-0.024	1050.000	47.143	346.000
$F$12	Daily dosage maximum	36.000	0.000	40.000	1E+30	4.000

Infinity

The shadow price for chemical X is positive, indicating that total cost increases as the requirement for chemical X increases.

The shadow price shows amount of decrease in total cost if chemical Z's limit is increased.

A: The reduced cost indicates that each ounce of ingredient B used to make the drink will cause the total cost per daily dosage to increase by \$0.069 ($\sim$\$0.07). The new cost will be \$13.06 + \$0.07 = \$13.13.

Alternatively, if Burn-Off can find a way of reducing ingredient B's cost per ounce by at least \$0.069 (to approximately \$0.13 or less per ounce), then it becomes cost-effective to use this ingredient to make the diet drink.

Q: There is some uncertainty in the cost of ingredient C. How sensitive is the current optimal solution to this cost?

A: The current cost of ingredient C is \$0.60 per ounce. The range for the cost coefficient of this ingredient shows an allowable increase of \$1.50 and an allowable decrease of \$0.073 in order for the current corner point solution to remain optimal. The cost per ounce of ingredient C could therefore fluctuate between \$0.527 (= \$0.60 − \$0.073) and \$2.10 (= \$0.60 + \$1.50) without affecting the current optimal mix.

The total cost will, however, change, depending on the actual unit cost of ingredient C. For example, if the cost of ingredient C increases to \$1.00 per ounce, the new total cost will be

$$= \$13.06 + (\$0.40 \text{ extra per ounce} \times 4.125 \text{ ounces of C})$$
$$= \$13.06 + \$1.65$$
$$= \$14.71$$

Q: What do the shadow prices for chemical X and chemical Z imply in this problem?

A: The shadow price for chemical X is \$0.088. Because the constraint for chemical X is a $\geq$ constraint, an increase by 1 unit in the RHS (from 280 to 281) makes the problem solution even more restrictive. That is, the feasible region becomes smaller. The optimal objective function value could, therefore, worsen. The shadow price indicates that for each additional unit of chemical X required to be present in the drink, the overall cost will increase by \$0.088. This value is valid for an increase of up to 41 units and a decrease by 11 units in the requirement for chemical X.

A negative value for shadow price implies that cost will decrease if the RHS value increases.

In contrast, the constraint for chemical Z is a $\leq$ constraint. An increase in the RHS of the constraint (from 1,050 to 1,051) will cause the feasible region to become bigger. Hence, the optimal objective function value could possibly improve. The negative value of the shadow price for this constraint indicates that each unit increase in the maximum limit allowed for chemical Z will cause the total cost to decrease by \$0.024. This value is valid for an increase of up to 47.143 units. Likewise, the total cost will *increase* by \$0.024 for each unit *decrease* in the maximum limit allowed for chemical Z. This is valid for a decrease of up to 346 units in the maximum limit for chemical Z.

Q: Burn-Off can decrease the minimum requirement for chemical X by 5 units (from 280 to 275), provided that the maximum limit allowed for chemical Z is reduced to 1,000 units (i.e., reduced by 50 units). Is this trade-off cost-effective for Burn-Off to implement?

Analyzing simultaneous changes requires the use of the 100% rule.

A: Because we are dealing with simultaneous changes in RHS values, we first verify whether the 100% rule is satisfied. To do so, we take the ratio of each proposed change to its maximum allowable change. The calculation is

$$\text{Sum of ratios} = (5/11) + (50/346) = 0.599 < 1$$

Because the sum does not exceed 1, we can use the shadow price information in the Sensitivity Report (Screenshot 4-4B). The reduction of 5 units in the requirement for chemical X will cause the feasible region to increase in size. The total cost will therefore improve (i.e., go down) by \$0.44 (= 5 units, at a shadow price of \$0.088 per unit).

In contrast, the reduction of 50 units in the maximum allowable limit for chemical Z makes the feasible region shrink in size. The total cost will therefore be adversely affected (i.e., go up) by \$1.20 (= 50 units, at a shadow price of \$0.024 per unit).

The net impact of this trade-off is therefore an increase in total cost of \$0.76 (= \$1.20 − \$0.44). The new cost will be \$13.82. Clearly, this trade-off is not cost-effective from Burn-Off's perspective and should be rejected.

Summary

In this chapter we present the important concept of sensitivity analysis. Sometimes referred to as postoptimality analysis, sensitivity analysis is used by management to answer a series of what-if questions about inputs to an LP model. It also tests just how sensitive the optimal solution is to changes in (1) objective function coefficients and (2) constraint RHS values.

We first explore sensitivity analysis graphically (i.e., for problems with only two decision variables). We then discuss how to interpret information in the Answer and Sensitivity Reports generated by Solver. We also discuss how the information in these reports can be used to analyze simultaneous changes in model parameter values and determine the potential impact of a new variable in the model.

Glossary

Allowable Decrease for an OFC The maximum amount by which the OFC of a decision variable can decrease for the current optimal solution to remain optimal.

Allowable Decrease for a RHS Value The maximum amount by which the RHS value of a constraint can decrease for the shadow price to be valid.

Allowable Increase for an OFC The maximum amount by which the OFC of a decision variable can increase for the current optimal solution to remain optimal.

Allowable Increase for a RHS Value The maximum amount by which the RHS value of a constraint can increase for the shadow price to be valid.

Answer Report A report created by Solver when it solves an LP model. This report presents the optimal solution in a detailed manner.

Objective Function Coefficient (OFC) The coefficient for a decision variable in the objective function. Typically, this refers to unit profit or unit cost.

100% Rule A rule used to verify the validity of the information in a Sensitivity Report when dealing with simultaneous changes to more than one RHS value or more than one OFC value.

Pricing Out A procedure by which the shadow price information in a Sensitivity Report can be used to gauge the impact of the addition of a new variable in an LP model.

Reduced Cost The difference between the marginal contribution to the objective function value from the inclusion of a decision variable and the marginal worth of the resources it consumes. In the case of a decision variable that has an optimal value of zero, it is also the minimum amount by which the OFC of that variable should change before it would have a nonzero optimal value.

Right-Hand Side (RHS) Value The amount of resource available (for a $\leq$ constraint) or the minimum requirement of some criterion (for a $\geq$ constraint). Typically expressed as a constant for sensitivity analysis.

Sensitivity Analysis The study of how sensitive an optimal solution is to model assumptions and to data changes. Also referred to as postoptimality analysis.

Shadow Price The magnitude of the change in the objective function value for a one-unit increase in the RHS of a constraint.

Slack The difference between the RHS and LHS of a $\leq$ constraint. Typically represents the unused resource.

Surplus The difference between the LHS and RHS of a $\geq$ constraint. Typically represents the level of oversatisfaction of a requirement.

Solved Problem

Solved Problem 4-1

Consider the Hong Kong Bank of Commerce and Industry example we first studied in Chapter 3 (section 3.5, on page 81). How can we use the Sensitivity Report for that example to detect the presence of alternate optimal solutions for an LP problem? Also, how can we use Excel to possibly identify those alternate optimal solutions?

Solution

For your convenience, we repeat the formulation portion of the Hong Kong Bank problem here. Define

F = number of full-time tellers to use (all starting at 9 A.M.)

P_1 = number of part-timers to use starting at 9 A.M. (leaving at 1 P.M.)

P_2 = number of part-timers to use starting at 10 A.M. (leaving at 2 P.M.)

P_3 = number of part-timers to use starting at 11 A.M. (leaving at 3 P.M.)

P_4 = number of part-timers to use starting at noon (leaving at 4 P.M.)

P_5 = number of part-timers to use starting at 1 P.M. (leaving at 5 P.M.)

Objective function:

$$\text{Minimize total daily personnel cost} = \$90F + \$28(P_1 + P_2 + P_3 + P_4 + P_5)$$

subject to the constraints:

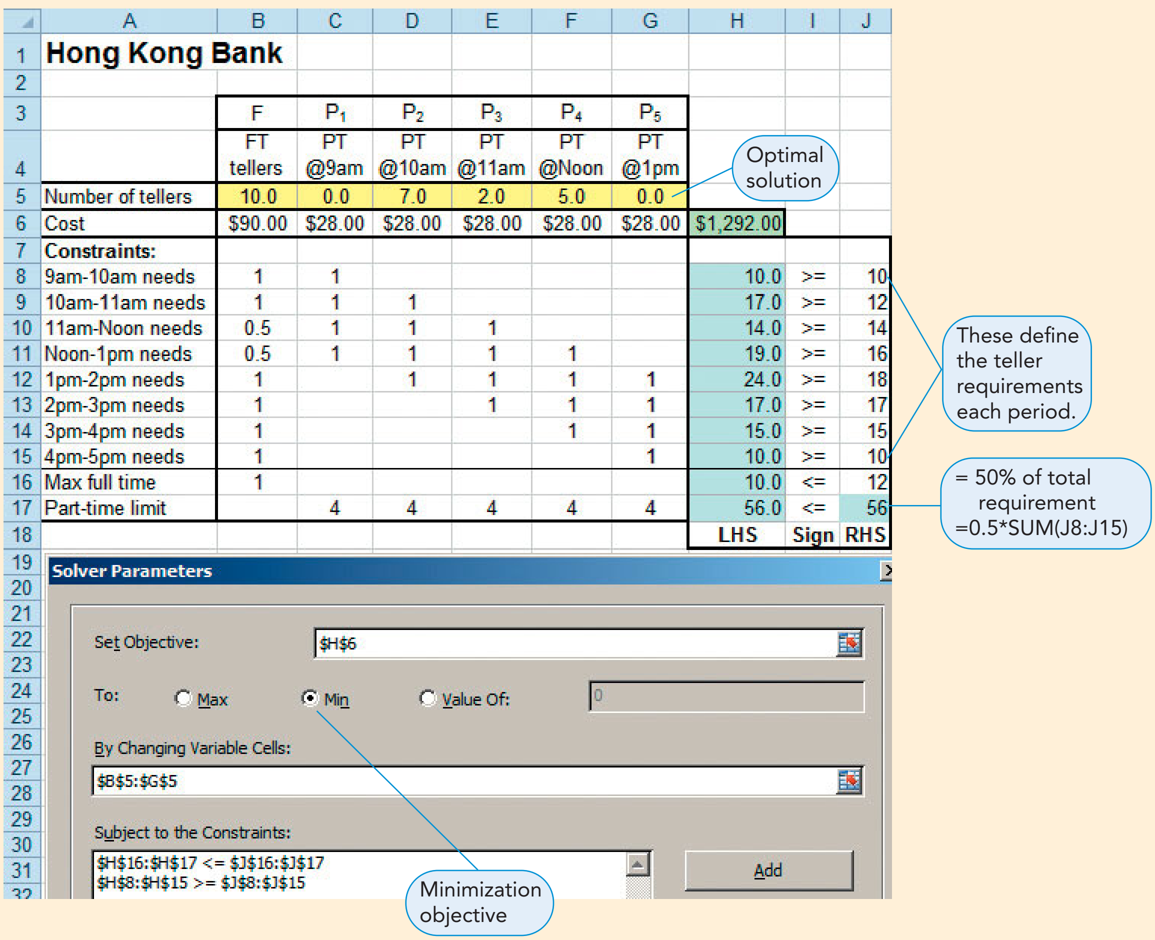
$$
\begin{array}{lll}
F + P_1 & \geq 10 & \text{(9 A.M.–10 A.M. requirement)} \\
F + P_1 + P_2 & \geq 12 & \text{(10 A.M.–11 A.M. requirement)} \\
0.5F + P_1 + P_2 + P_3 & \geq 14 & \text{(11 A.M.–12 noon requirement)} \\
0.5F + P_1 + P_2 + P_3 + P_4 & \geq 16 & \text{(12 noon–1 P.M. requirement)} \\
F + P_2 + P_3 + P_4 + P_5 & \geq 18 & \text{(1 P.M.–2 P.M. requirement)} \\
F + P_3 + P_4 + P_5 & \geq 17 & \text{(2 P.M.–3 P.M. requirement)} \\
F + P_4 + P_5 & \geq 15 & \text{(3 P.M.–4 P.M. requirement)} \\
F + P_5 & \geq 10 & \text{(4 P.M.–5 P.M. requirement)} \\
F & \leq 12 & \text{(full-time tellers)} \\
4P_1 + 4P_2 + 4P_3 + 4P_4 + 4P_5 & \leq 56 & \text{(part-time workers limit)} \\
F, P_1, P_2, P_3, P_4, P_5 & \geq 0 & \text{(nonnegativity)}
\end{array}
$$

WWW

File: 4-5.xls

The Excel layout and Solver entries for this model are shown in Screenshot 4-5A. The Sensitivity Report is shown in Screenshot 4-5B.

SCREENSHOT 4-5A Excel Layout and Solver Entries for Hong Kong Bank



	A	B	C	D	E	F	G	H	I	J
1	Hong Kong Bank									
2										
3		F	P_1	P_2	P_3	P_4	P_5			
4		FT tellers	PT @9am	PT @10am	PT @11am	PT @Noon	PT @1pm			
5	Number of tellers	10.0	0.0	7.0	2.0	5.0	0.0			
6	Cost	$90.00	$28.00	$28.00	$28.00	$28.00	$28.00	$1,292.00		
7	Constraints:									
8	9am-10am needs	1	1					10.0	>=	10
9	10am-11am needs	1	1	1				17.0	>=	12
10	11am-Noon needs	0.5	1	1	1			14.0	>=	14
11	Noon-1pm needs	0.5	1	1	1	1		19.0	>=	16
12	1pm-2pm needs	1		1	1	1	1	24.0	>=	18
13	2pm-3pm needs	1			1	1	1	17.0	>=	17
14	3pm-4pm needs	1				1	1	15.0	>=	15
15	4pm-5pm needs	1					1	10.0	>=	10
16	Max full time	1						10.0	<=	12
17	Part-time limit		4	4	4	4	4	56.0	<=	56
18								LHS	Sign	RHS

SCREENSHOT 4-5B Solver Sensitivity Report for Hong Kong Bank

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [4-5.xls]Hong Kong Bank

Reduced cost of zero indicates that a solution that uses part-time tellers at 9 A.M. exists, with no change in the optimal objective value.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number of tellers FT tellers	10.00	0.00	90.00	1E+30	48.00
$C$5	Number of tellers PT @9am	0.00	0.00	28.00	48.00	0.00
$D$5	Number of tellers PT @10am	7.00	0.00	28.00	0.00	45.00
$E$5	Number of tellers PT @11am	2.00	0.00	28.00	60.00	0.00
$F$5	Number of tellers PT @Noon	5.00	0.00	28.00	0.00	60.00
$G$5	Number of tellers PT @1pm	0.00	0.00	28.00	48.00	0.00

These zeros indicate that there are alternate optimal solutions.

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$8	9am-10am needs	10.00	0.00	10.00	6.00	0.00
$H$9	10am-11am needs	17.00	0.00	12.00	5.00	1E+30
$H$10	11am-Noon needs	14.00	60.00	14.00	0.00	3.00
$H$11	Noon-1pm needs	19.00	0.00	16.00	3.00	1E+30
$H$12	1pm-2pm needs	24.00	0.00	18.00	6.00	1E+30
$H$13	2pm-3pm needs	17.00	0.00	17.00	5.00	2.00
$H$14	3pm-4pm needs	15.00	60.00	15.00	0.00	3.00
$H$15	4pm-5pm needs	10.00	0.00	10.00	3.00	0.00
$H$16	Max full time	10.00	0.00	12.00	1E+30	2.00
$H$17	Part-time limit	56.00	-8.00	56.00	60.00	0.00

Screenshot 4-5A reveals that the optimal solution is to employ 10 full-time tellers, 7 part-time tellers at 10 A.M., 2 part-time tellers at 11 A.M., and 5 part-time tellers at noon, for a total cost of $1,292 per day.

In Screenshot 4-5B, the shadow price of −$8 for the part-time limit of 56 hours indicates that each additional hour (over the 56-hour limit) that part-time tellers are allowed to work will allow the bank to reduce costs by $8. This shadow price is valid for a limit of 60 more hours (i.e., up to 116 hours).

Examining the Allowable Increase and Allowable Decrease columns for the OFCs, we see that there are several values of zero in these columns. This indicates that there are alternate optimal solutions to this problem.

Likewise, consider the reduced cost for variables P_1 (part-timers starting at 9 A.M.) and P_5 (part-timers starting at 1 P.M.). These are zero, even though these variables have values of zero (their lower limit). This implies that, for example, it is possible to force P_1 (or P_5) to have a nonzero value at optimality and not affect the total cost in any way. This is another indication of the presence of alternate optimal solutions to this problem.

We can force Excel to identify alternate optimal solutions.

How can we identify these optimal solutions by using Excel's Solver? There are at least a couple ways of doing so. First, simply rearranging the order in which the variables and/or constraints are included in the Excel layout may make Solver identify an alternate optimal solution. That is, we can just swap the order in which some of the rows and/or columns are included in the model. There is, however, no guarantee that this approach will always identify an alternate optimal solution.

The second approach, which will definitely find an alternate optimal solution (if one exists), is as follows. From the preceding discussion, we know that variable P_1 (which currently has a zero value) can have a nonzero value at an optimal solution. To force this to happen, we include the current objective function as a constraint, as follows:

$$\$90F + \$28(P_1 + P_2 + P_3 + P_4 + P_5) = \$1{,}292$$

This will force the new solution to have the same optimal cost (i.e., it is also an optimal solution). Then, the new objective for the model would be

$$\text{Max } P_1$$

Note that this will find a solution that costs \$1,292 but has a nonzero value for P_1. We can repeat the same approach with the variable P_5 to find yet another optimal solution.

Using these approaches, we can identify two alternate solutions for Hong Kong Bank, as follows:

1. 10 full-time tellers, 6 part-time tellers at 9 A.M., 1 part-time teller at 10 A.M., 2 part-time tellers at 11 A.M., and 5 part-time tellers at noon.
2. 10 full-time tellers, 6 part-time tellers at 9 A.M., 1 part-time teller at 10 A.M., 2 part-time tellers at 11 A.M., 2 part-time tellers at noon, and 3 part-time tellers at 10 A.M.

The cost of each of these employment policies is also \$1,292.

Discussion Questions and Problems

Discussion Questions

4-1 Discuss the role of sensitivity analysis in LP. Under what circumstances is it needed, and under what conditions do you think it is not necessary?

4-2 Is sensitivity analysis a concept applied to LP only, or should it also be used when analyzing other techniques (e.g., break-even analysis)? Provide examples to prove your point.

4-3 Explain how a change in resource availability can affect the optimal solution of a problem.

4-4 Explain how a change in an OFC can affect the optimal solution of a problem.

4-5 Are simultaneous changes in input data values logical? Provide examples to prove your point.

4-6 Explain the 100% rule and its role in analyzing the impact of simultaneous changes in model input data values.

4-7 How can a firm benefit from using the pricing out procedure?

4-8 How do we detect the presence of alternate optimal solutions from a Solver Sensitivity Report?

4-9 Why would a firm find information regarding the shadow price of a resource useful?

Problems

4-10 A graphical approach was used to solve the following LP model in Problem 2-14:

$$\text{Maximize profit} = \$4X + \$5Y$$

Subject to the constraints

$$\begin{aligned} 5X + 2Y &\le 40 \\ 3X + 6Y &\le 30 \\ X &\le 7 \\ 2X - Y &\ge 3 \\ X, Y &\ge 0 \end{aligned}$$

Use the graphical solution to answer the following questions. Each question is independent of the others. Determine if (and how) the following changes would affect the optimal solution values and/or profit.

(a) A technical breakthrough raises the profit per unit of Y to \$10.
(b) The profit per unit of X decreases to only \$2.
(c) The first constraint changes to $5X + 2Y \le 54$.

4-11 A graphical approach was used to solve the following LP model in Problem 2-16:

$$\text{Minimize cost} = \$4X + \$7Y$$

Subject to the constraints

$$\begin{aligned} 2X + 3Y &\ge 60 \\ 4X + 2Y &\ge 80 \\ X &\le 24 \\ X, Y &\ge 0 \end{aligned}$$

Use the graphical solution to answer the following questions. Each question is independent of the others. Determine if (and how) the following changes would affect the optimal solution values and/or cost.

(a) The cost per unit of Y increases to \$9.
(b) The first constraint changes to $2X + 3Y \ge 90$.
(c) The third constraint changes to $X \le 15$.

4-12 A graphical approach was used to solve the following LP model in Problem 2-15:

$$\text{Maximize profit} = \$4X + \$3Y$$

Subject to the constraints

$$\begin{aligned} 2X + 4Y &\le 72 \\ 3X + 6Y &\ge 27 \\ -3X + 10Y &\ge 0 \\ X, Y &\ge 0 \end{aligned}$$

Use the graphical solution to answer the following questions. Each question is independent of the others. Determine if (and how) the following changes would affect the optimal solution values and/or profit.

(a) The profit per unit of X decreases to \$1.
(b) The first constraint changes to $2X + 4Y \leq 80$.
(c) The third constraint changes to $-3X + 10Y \leq 0$.

4-13 Consider the Win Big Gambling Club media selection example discussed in section 3.3 (page 73) of Chapter 3. Use the Sensitivity Report for this LP model (shown in Screenshot 4-6) to answer the following questions. Each question is independent of the others.

What is the impact on the audience coverage under the following scenarios?

(a) Management approves spending \$200 more on radio advertising each week.
(b) The contractual agreement to place at least five radio spots per week is eliminated.
(c) The audience reached per ad increases to 3,100.
(d) There is some uncertainty in the audience reached per TV spot. For what range of values for this OFC will the current solution remain optimal?

4-14 Consider the MSA marketing research example discussed in Section 3.3 (page 74) of Chapter 3. Use the Sensitivity Report for this LP model (shown in Screenshot 4-7 on page 150) to answer the following questions. Each question is independent of the others.

(a) What is the maximum unit cost that will make it worthwhile to include in the survey persons 30 years of age or younger who live in a border state?
(b) What is the impact if MSA wants to increase the sample size to 3,000?
(c) What is the impact if MSA insists on including people 31–50 years of age who do not live in a border state?
(d) What is the impact if we can reduce the minimum 30 or younger persons required to 900, provided that we raise the persons 31–50 years of age to 650?

4-15 Consider the Whole Food Nutrition Center diet problem example discussed in Section 3.7 (page 89) of Chapter 3. Use the Sensitivity Report for this LP model (shown in Screenshot 4-8 on page 150) to answer the following questions. Each question is independent of the others.

(a) What is the impact if the daily allowance for protein can be reduced to 2.9 units?
(b) Whole Food believes the unit price of grain A could be 5% overestimated and the unit price of grain B could be 10% underestimated. If these turn out to be true, what is the new optimal solution and optimal total cost?
(c) What is the impact if the reduction in the daily allowance for protein in (a) requires Whole Food to simultaneously increase the daily allowance of riboflavin to 2.20 units?

4-16 Consider the cell phone manufacturing problem presented in Chapter 3 as Problem 3-6 on page 105. Use Solver to create the Sensitivity Report for this LP problem. Use this report to answer the following questions. Each question is independent of the others.

(a) Interpret the reduced costs for the products that are not currently included in the optimal production plan.

SCREENSHOT 4-6 Solver Sensitivity Report for Problem 4-13: Win Big Gambling Club

Microsoft Excel 14.0 Sensitivity Report
Problem 4-13. Win Big Gambling Club

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$5	Number of units TV spots	1.97	0.00	5000.00	1620.69	5000.00
\$C\$5	Number of units Newspaper ads	5.00	0.00	8500.00	1E+30	2718.75
\$D\$5	Number of units Prime-time radio spots	6.21	0.00	2400.00	1E+30	263.16
\$E\$5	Number of units Afternoon radio spots	0.00	-344.83	2800.00	344.83	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$F\$8	Maximum TV	1.97	0.00	12.00	1E+30	10.03
\$F\$9	Maximum newspaper	5.00	2718.75	5.00	1.70	5.00
\$F\$10	Max prime-time radio	6.21	0.00	25.00	1E+30	18.79
\$F\$11	Max afternoon radio	0.00	0.00	20.00	1E+30	20.00
\$F\$12	Total budget	8,000.00	6.25	8000.00	8025.00	1575.00
\$F\$13	Maximum radio \$	1,800.00	2.03	1800.00	1575.00	350.00
\$F\$14	Minimum radio spots	6.21	0.00	5.00	1.21	1E+30

SCREENSHOT 4-7 Solver Sensitivity Report for Problem 4-14: MSA Marketing Research

Microsoft Excel 14.0 Sensitivity Report
Problem 4-14. Management Science Associates

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number of households <= 30 and border	0.00	0.60	7.50	1E+30	0.60
$C$5	Number of households 31-50 and border	600.00	0.00	6.80	0.45	0.82
$D$5	Number of households >= 51 and border	140.00	0.00	5.50	0.6	29.90
$E$5	Number of households <= 30 and not border	1000.00	0.00	6.90	0.6	0.92
$F$5	Number of households 31-50 and not border	0.00	0.45	7.25	1E+30	0.45
$G$5	Number of households >= 51 and not border	560.00	0.00	6.10	1.025	0.60

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$8	Total households	2300.00	5.98	2300.00	1E+30	700.00
$H$9	<= 30 households	1000.00	0.92	1000.00	700.00	1000.00
$H$10	31-50 households	600.00	0.82	600.00	700.00	493.75
$H$11	Border Mexico	740.00	0.00	0.00	395.00	1E+30
$H$12	<= 30 and not border	1000.00	0.00	0.00	500.00	1E+30
$H$13	>= 51 and border	140.00	-0.60	0.00	560.00	140.00

(b) Another part of the corporation wants to take 35 hours of time on test device 3. How does this affect the optimal solution?

(c) The company has the opportunity to obtain 20 additional hours on test device 1 at a cost of $25 per hour. Would this be worthwhile?

(d) The company has the opportunity to give up 20 hours of time on device 1 and obtain 40 hours of time on device 2 in return. Would this be worthwhile? Justify your answer.

4-17 Consider the family farm planning problem presented in Chapter 3 as Problem 3-27 on page 110. Use Solver to create the Sensitivity Report for this LP problem. Use this report to answer the following questions. Each question is independent of the others.

(a) Is this solution a unique optimal solution? Why or why not?

(b) If there are alternate solutions, use Solver to identify at least one other optimal solution.

(c) Would the total profit increase if barley sales could be increased by 10%? If so, by how much?

(d) Would the availability of more water increase the total profit? If so, by how much?

4-18 Consider the boarding stable feed problem presented in Chapter 3 as Problem 3-32 on page 112. Use Solver to create the Sensitivity Report for this LP problem. Use this report to answer the

SCREENSHOT 4-8 Solver Sensitivity Report for Problem 4-15: Whole Food Nutrition Center

Microsoft Excel 14.0 Sensitivity Report
Problem 4-15. Whole Food Nutrition Center

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number of pounds Grain A	0.025	0.000	0.330	0.063	1E+30
$C$5	Number of pounds Grain B	0.050	0.000	0.470	1E+30	0.190
$D$5	Number of pounds Grain C	0.050	0.000	0.380	1E+30	0.073

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$8	Protein	3.000	0.038	3.000	0.000	0.250
$E$9	Riboflavin	2.350	0.000	2.000	0.350	1E+30
$E$10	Phosphorus	1.000	0.088	1.000	0.018	0.000
$E$11	Magnesium	0.425	0.000	0.425	0.000	1E+30
$E$12	Total Mix	0.125	-1.210	0.125	0.004	0.000

following questions. Each question is independent of the others.

(a) If the price of grain decreases by \$0.01 per pound, will the optimal solution change?
(b) Which constraints are binding? Interpret the shadow price for the binding constraints.
(c) What would happen to the total cost if the price of mineral decreased by 20% from its current value?
(d) For what price range of oats is the current solution optimal?

4-19 Consider the campus dietitian's problem presented in Chapter 3 as Problem 3-33 on page 112. Use Solver to create the Sensitivity Report for this LP problem. Use this report to answer the following questions. Each question is independent of the others.

(a) Interpret the shadow prices for the carbohydrates and iron constraints.
(b) What would happen to total cost if the dietitian chooses to use milk in the diet?
(c) What would be the maximum amount the dietitian would be willing to pay for beans to make it a cost-effective item for inclusion in the diet?
(d) Is the solution to this problem a unique optimal solution? Justify your answer.

4-20 Consider the following LP problem, in which X and Y denote the number of units of products X and Y to produce, respectively:

$$\text{Maximize profit} = \$4X + \$5Y$$

subject to the constraints

$$\begin{aligned} X + 2Y &\leq 10 && \text{(labor available, in hours)} \\ 6X + 6Y &\leq 36 && \text{(material available, in pounds)} \\ 8X + 4Y &\leq 40 && \text{(storage available, in square feet)} \\ X, Y &\geq 0 && \text{(nonnegativity)} \end{aligned}$$

The Excel Sensitivity Report for this problem is shown in Screenshot 4-9. Calculate and explain what happens to the optimal solution for each of the following situations. Each question is independent of the other questions.

(a) You acquire 2 additional pounds of material.
(b) You acquire 1.5 additional hours of labor.
(c) You give up 1 hour of labor and get 1.5 pounds of material.
(d) The profit contributions for both products X and Y are changed to \$4.75 each.
(e) You decide to introduce a new product that has a profit contribution of \$2. Each unit of this product will use 1 hour of labor, 1 pound of material, and 2 square feet of storage space.

4-21 Consider the following LP problem, in which X and Y denote the number of units of products X and Y to produce, respectively:

$$\text{Maximize profit} = \$5X + \$5Y$$

subject to the constraints

$$\begin{aligned} 2X + 3Y &\leq 60 && \text{(resource 1)} \\ 4X + 2Y &\leq 80 && \text{(resource 2)} \\ X \quad\quad &\leq 18 && \text{(resource 3)} \\ X, Y &\geq 0 && \text{(nonnegativity)} \end{aligned}$$

The Excel Sensitivity Report for this problem is shown in Screenshot 4-10 on page 152. Calculate and explain what happens to the optimal solution for each of the following situations. Each question is independent of the other questions.

(a) What is the optimal solution to this problem?
(b) For what ranges of values, holding all else constant, could each of the objective function coefficients be changed without changing the optimal solution?
(c) If we could obtain one additional unit of resource 1, how would it impact profit? Over what range of RHS values could we rely upon this value?
(d) If we were to give up one unit of resource 2, how would it impact profit? Over what range of RHS values could we rely upon this value?

SCREENSHOT 4-9
Solver Sensitivity Report for Problem 4-20

Microsoft Excel 14.0 Sensitivity Report
Problem 4-20

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$4	Solution value X	2.00	0.00	4.00	1.00	1.50
\$C\$4	Solution value Y	4.00	0.00	5.00	3.00	1.00

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$D\$7	Labor	10.00	1.00	10.00	2.00	2.00
\$D\$8	Material	36.00	0.50	36.00	4.00	6.00
\$D\$9	Storage	32.00	0.00	40.00	1E+30	8.00

SCREENSHOT 4-10
Solver Sensitivity Report for Problem 4-21

Microsoft Excel 14.0 Sensitivity Report
Problem 4-21

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$4	Solution value X	15.00	0.00	5.00	5.00	1.67
$C$4	Solution value Y	10.00	0.00	5.00	2.50	2.50

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$7	Resource 1	60.00	1.25	60.00	60.00	12.00
$D$8	Resource 2	80.00	0.63	80.00	8.00	40.00
$D$9	Resource 3	15.00	0.00	18.00	1E+30	3.00

(e) If we were to increase the profit for product X by $4, how would the solution change? What would be the new solution values and the new profit?
(f) If we were to decrease the profit for product Y by $2, how would the solution change? What would be the new solution values and the new profit?
(g) Suppose that two units of resource 3 were found to be unusable, how would the solution change?
(h) If we were able to obtain five more units of resource 3, would you be interested in the deal? Why or why not?
(i) If the profit for product X was increased to $7 while at the same time the profit for product Y was reduced to $4, what would be the new solution values and the new profit?
(j) Suppose you were to be offered 50 units of resource 1 at a premium of $1 each over the existing cost price for that resource. Would you purchase this? If so, by how much would your profit increase?

4-22 The Good-to-Go Suitcase Company makes three kinds of suitcases: (1) Standard, (2) Deluxe, and (3) Luxury styles. Each suitcase goes through four production stages: (1) cutting and coloring, (2) assembly, (3) finishing, and (4) quality and packaging. The total number of hours available in each of these departments is 630, 600, 708, and 135, respectively.

Each Standard suitcase requires 0.7 hours of cutting and coloring, 0.5 hours of assembly, 1 hour of finishing, and 0.1 hours of quality and packaging. The corresponding numbers for each Deluxe suitcase are 1 hour, 5/6 hours, 2/3 hours, and 0.25 hours, respectively. Likewise, the corresponding numbers for each Luxury suitcase are 1 hour, 2/3 hours, 0.9 hours, and 0.4 hours, respectively.

The sales revenue for each type of suitcase is as follows: Standard $36.05, Deluxe $39.50, and Luxury $43.30. The material costs are Standard $6.25, Deluxe $7.50, and Luxury $8.50. The hourly cost of labor for each department is cutting and coloring $10, assembly $6, finishing $9, and quality and packaging $8.

The Excel layout and LP Sensitivity Report of Good-to-Go's problem are shown in Screenshots 4-11A and 4-11B, respectively. Each of the following questions is independent of the others.

SCREENSHOT 4-11A
Excel Layout for Good-to-Go Suitcase Company

	A	B	C	D	E	F	G	H
1	Good-to-Go Suitcase Company							
2								
3		Standard	Deluxe	Luxury				
4	Solution value	540.00	252.00	0.00				
5	Selling price per unit	$36.05	$39.50	$43.30	$29,421.00			
6	Material cost per unit	$6.25	$7.50	$8.50	$5,265.00			
7	Labor cost per unit	$19.80	$23.00	$25.30	$16,488.00			
8	Profit	$10.00	$9.00	$9.50	$7,668.00			
9	Constraints							Cost
10	Cutting & Coloring	0.70	1.00	1.00	630.00	<=	630	$10
11	Assembly	0.50	0.83	0.67	480.00	<=	600	$6
12	Finishing	1.00	0.67	0.90	708.00	<=	708	$9
13	Quality & Packaging	0.10	0.25	0.40	117.00	<=	135	$8
14					LHS	Sign	RHS	

SCREENSHOT 4-11B
Solver Sensitivity Report for Good-to-Go Suitcase Company

Microsoft Excel 14.0 Sensitivity Report
Problems P4-22&23. Good-to-Go Suitcase Company

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$4	Solution value Standard	540.00	0.00	10.00	3.50	2.56
$C$4	Solution value Deluxe	252.00	0.00	9.00	5.29	1.61
$D$4	Solution value Luxury	0.00	-1.12	9.50	1.12	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$10	Cutting & Coloring	630.00	4.38	630	52.36	134.40
$E$11	Assembly	480.00	0.00	600	1E+30	120.00
$E$12	Finishing	708.00	6.94	708	192.00	128.00
$E$13	Quality & Packaging	117.00	0.00	135	1E+30	18.00

(a) What is the optimal production plan? Which of the resources are scarce?

(b) Suppose Good-to-Go is considering including a polishing process, the cost of which would be added directly to the price. Each Standard suitcase would require 10 minutes of time in this treatment, each Deluxe suitcase would need 15 minutes, and each Luxury suitcase would need 20 minutes. Would the current production plan change as a result of this additional process if 170 hours of polishing time were available? Explain your answer.

(c) Now consider the addition of a waterproofing process where each Standard suitcase would use 1 hour of time in the process, each Deluxe suitcase would need 1.5 hours, and each Luxury suitcase would require 1.75 hours. Would this change the production plan if 900 hours were available? Why or why not?

Source: Professors Mark and Judith McKnew, Clemson University.

4-23 Suppose Good-to-Go (Problem 4-22) is considering the possible introduction of two new products to its line of suitcases: the Compact model (for teenagers) and the Kiddo model (for children). Market research suggests that Good-to-Go can sell the Compact model for no more than \$30, whereas the Kiddo model would go for as much as \$37.50 to specialty toy stores. The amount of labor and the cost of raw materials for each possible new product are as follows:

COST CATEGORY	COMPACT	KIDDO
Cutting and coloring (hr.)	0.50	1.20
Assembly (hr.)	0.75	0.75
Finishing (hr.)	0.75	0.50
Quality and packaging (hr.)	0.20	0.20
Raw materials	\$5.00	\$4.50

Use a pricing out strategy to check if either model would be economically attractive to make.

4-24 The Strollers-to-Go Company makes lightweight umbrella-type strollers for three different groups of children. The TiniTote is designed specifically for newborns who require extra neck support. The ToddleTote is for toddlers up to 30 pounds. Finally, the company produces a heavy-duty model called TubbyTote, which is designed to carry children up to 60 pounds. The stroller company is in the process of determining its production for each of the three types of strollers for the upcoming planning period.

The marketing department has forecast the following maximum demand for each of the strollers during the planning period: TiniTote 180, TubbyTote 70, and ToddleTote 160. Strollers-to-Go sells TiniTotes for \$63.75, TubbyTotes for \$82.50, and ToddleTotes for \$66. As a matter of policy, it wants to produce no less than 50% of the forecast demand for each product. It also wants to keep production of ToddleTotes to a maximum of 40% of total stroller production.

The production department has estimated that the material costs for TiniTote, TubbyTote, and ToddleTote strollers will be \$4, \$6, and \$5.50 per unit, respectively. The strollers are processed through fabrication, sewing, and assembly workstations. The metal and plastic frames are made in the fabrication station. The fabric seats are cut and stitched together in the sewing station. Finally, the frames are put together with the seats in the assembly station. In the upcoming planning period, there will be 620 hours available in fabrication, where the direct labor cost is \$8.25 per hour. The sewing station has 500 hours available, and the direct labor cost is \$8.50 per hour. The assembly station has 480 hours available, and the direct labor cost is \$8.75 per hour.

The standard processing rate for TiniTotes is 3 hours in fabrication, 2 hours in sewing, and

SCREENSHOT 4-12A
Excel Layout for Strollers-to-Go Company

	A	B	C	D	E	F	G	H
1	Strollers-to-Go Company							
2								
3		TiniTote	TubbyTote	ToddleTote				
4	Solution value	100.00	35.00	90.00				
5	Selling price per unit	$63.75	$82.50	$66.00	$15,202.50			
6	Material cost per unit	$4.00	$6.00	$5.50	$1,105.00			
7	Labor cost per unit	$50.50	$67.75	$51.00	$12,011.25			
8	Profit	$9.25	$8.75	$9.50	$2,086.25			
9	Constraints							Cost
10	Fabrication	3.0	4.0	2.0	620.00	<=	620	$8.25
11	Sewing	2.0	1.0	2.0	415.00	<=	500	$8.50
12	Assembly	1.0	3.0	2.0	385.00	<=	480	$8.75
13	Tinitote demand	1.0			100.00	<=	180	
14	Tubbytote demand		1.0		35.00	<=	70	
15	Toddletote demand			1.0	90.00	<=	160	
16	Toddletote max prod ratio	-0.4	-0.4	0.6	0.00	<=	0	
17	Tinitote min prod	1.0			100.00	>=	90	
18	Tubbytote min prod		1.0		35.00	>=	35	
19	Toddletote min prod			1.0	90.00	>=	80	
20					LHS	Sign	RHS	

1 hour in assembly. TubbyTotes require 4 hours in fabrication, 1 hour in sewing and 3 hours in assembly, whereas ToddleTotes require 2 hours in each station.

The Excel layout and LP Sensitivity Report for Strollers-to-Go's problem are shown in Screenshots 4-12A and 4-12B, respectively. Each of the following questions is independent of the others.

(a) How many strollers of each type should Strollers-to-Go make? What is the profit? Which constraints are binding?
(b) How much labor time is being used in the fabrication, sewing, and assembly areas?
(c) How much would Strollers-to-Go be willing to pay for an additional hour of fabrication time? For an additional hour of sewing time?
(d) Is Strollers-to-Go producing any product at its maximum sales level? Is it producing any product at its minimum level?

Source: Professors Mark and Judith McKnew, Clemson University.

4-25 Consider the Strollers-to-Go production problem (Problem 4-24).
(a) Over what range of costs could the TiniTote materials vary and the current production plan

SCREENSHOT 4-12B
Solver Sensitivity Report for Strollers-to-Go Company

Microsoft Excel 14.0 Sensitivity Report
Problems 4-24to27. Strollers-to-Go Company

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$4	Solution value TiniTote	100.00	0.00	9.25	5.00	3.33
$C$4	Solution value TubbyTote	35.00	0.00	8.75	4.10	1E+30
$D$4	Solution value ToddleTote	90.00	0.00	9.50	1E+30	3.33

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$10	Fabrication	620.00	3.60	620.00	110.50	43.33
$E$11	Sewing	415.00	0.00	500.00	1E+30	85.00
$E$12	Assembly	385.00	0.00	480.00	1E+30	95.00
$E$13	Tinitote demand	100.00	0.00	180.00	1E+30	80.00
$E$14	Tubbytote demand	35.00	0.00	70.00	1E+30	35.00
$E$15	Toddletote demand	90.00	0.00	160.00	1E+30	70.00
$E$16	Toddletote max prod ratio	0.00	3.85	0.00	13.00	8.67
$E$17	Tinitote min prod	100.00	0.00	90.00	10.00	1E+30
$E$18	Tubbytote min prod	35.00	-4.10	35.00	8.13	35.00
$E$19	Toddletote min prod	90.00	0.00	80.00	10.00	1E+30

remain optimal? (*Hint*: How are material costs reflected in the problem formulation?)

(b) Suppose that Strollers-to-Go decided to polish each stroller prior to shipping. The process is fast and would require 10, 15, and 12 minutes, respectively, for TiniTote, TubbyTote, and ToddleTote strollers. Would this change the current production plan if 48 hours of polishing time were available?

4-26 Consider the Strollers-to-Go production problem (Problem 4-24).

(a) Suppose that Strollers-to-Go could purchase additional fabrication time at a cost of \$10.50 per hour. Should it be interested? Why or why not? What is the most that it would be willing to pay for an additional hour of fabrication time?

(b) Suppose that Strollers-to-Go could only purchase fabrication time in multiples of 40-hour bundles. How many bundles should it be willing to purchase then?

4-27 Suppose that Strollers-to-Go (Problem 4-24) is considering the production of TwinTotes for families who are doubly blessed. Each TwinTote would require \$7.10 in materials, 4 hours of fabrication time, 2 hours of sewing time, and 2 hours to assemble. Would this product be economically attractive to manufacture if the sales price were \$86? Why or why not?

4-28 The Classic Furniture Company is trying to determine the optimal quantities to make of six possible products: tables and chairs made of oak, cherry, and pine. The products are to be made using the following resources: labor hours and three types of wood. Minimum production requirements are as follows: at least 3 each of oak and cherry tables, at least 10 each of oak and cherry chairs, and at least 5 pine chairs.

The Excel layout and LP Sensitivity Report for Classic Furniture's problem are shown in Screenshots 4-13A and 4-13B, respectively. The objective function coefficients in the Screenshots refer to unit profit per item. Each of the following questions is independent of the others.

(a) What is the profit represented by the objective function, and what is the production plan?

(b) Which constraints are binding?

(c) What is the range over which the unit profit for oak chairs can change without changing the production plan?

(d) What is the range over which the amount of available oak could range without changing the combination of binding constraints?

(e) Does this Sensitivity Report indicate the presence of multiple optima? How do you know?

(f) After production is over, how many pounds of cherry wood will be left over?

(g) According to this report, how many more chairs were made than were required?

4-29 Consider the Classic Furniture product mix problem (Problem 4-28). For each of the following situations, what would be the impact on the production plan and profit? If it is possible to compute the new profit or production plan, do so.

(a) The unit profit for oak tables increases to \$83.

(b) The unit profit for pine chairs decreases to \$13.

(c) The unit profit for pine tables increases by \$20.

(d) The unit profit for cherry tables decreases to \$85.

(e) The company is required to make at least 20 pine chairs.

(f) The company is required to make no more than 55 cherry chairs.

4-30 Consider the Classic Furniture product mix problem (Problem 4-28). For each of the following situations,

SCREENSHOT 4-13A
Excel Layout for Classic Furniture Company

	A	B	C	D	E	F	G	H	I	J
1	**Classic Furniture Company**									
2										
3		Oak tables	Oak chairs	Cherry tables	Cherry chairs	Pine tables	Pine chairs			
4	Number of units	3.00	51.67	3.00	85.56	42.26	33.08			
5	Profit	\$75	\$35	\$90	\$60	\$45	\$20	\$10,000.00		
6	**Constraints**									
7	Labor hours	7.5	3.5	9.0	6.0	4.5	2.0	1000.00	<=	1,000
8	Oak (pounds)	200	30					2150.00	<=	2,150
9	Cherry (pounds)			240	36			3800.00	<=	3,800
10	Pine (pounds)					180	27	8500.00	<=	8,500
11	Min oak tables	1						3.00	>=	3
12	Min cherry tables			1				3.00	>=	3
13	Min oak chairs		1					51.67	>=	10
14	Min cherry chairs				1			85.56	>=	10
15	Min pine chairs						1	33.08	>=	5
16								**LHS**	**Sign**	**RHS**

SCREENSHOT 4-13B
Solver Sensitivity Report for Classic Furniture Company

Microsoft Excel 14.0 Sensitivity Report
Problems 4-28to32. Classic Furniture Company

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$4	Number of units Oak tables	3.00	0.00	75.00	0.00	1E+30
$C$4	Number of units Oak chairs	51.67	0.00	35.00	1E+30	0.00
$D$4	Number of units Cherry tables	3.00	0.00	90.00	0.00	1E+30
$E$4	Number of units Cherry chairs	85.56	0.00	60.00	1E+30	0.00
$F$4	Number of units Pine tables	42.26	0.00	45.00	88.33	0.00
$G$4	Number of units Pine chairs	33.08	0.00	20.00	0.00	13.25

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$7	Labor hours	1000.00	10.00	1000.00	373.30	37.21
$H$8	Oak (pounds)	2150.00	0.00	2150.00	318.93	1250.00
$H$9	Cherry (pounds)	3800.00	0.00	3800.00	223.25	2239.78
$H$10	Pine (pounds)	8500.00	0.00	8500.00	1488.33	5039.50
$H$11	Min oak tables	3.00	0.00	3.00	6.25	2.35
$H$12	Min cherry tables	3.00	0.00	3.00	11.33	1.20
$H$13	Min oak chairs	51.67	0.00	10.00	41.67	1E+30
$H$14	Min cherry chairs	85.56	0.00	10.00	75.56	1E+30
$H$15	Min pine chairs	33.08	0.00	5.00	28.08	1E+30

what would be the impact on the production plan and profit? If it is possible to compute the new profit or production plan, do so.

(a) The number of labor hours expands to 1,320.
(b) The amount of cherry wood increases to 3,900.
(c) The number of labor hours decreases to 950.
(d) The company does not have a minimum requirement for cherry chairs.

4-31 Consider the Classic Furniture product mix problem (Problem 4-28). For each of the following situations, what would be the impact on the production plan and profit? If it is possible to compute the new profit or production plan, do so.

(a) OFCs for oak tables and cherry tables each decreases by $15.
(b) OFCs for oak tables and oak chairs are reversed.
(c) OFCs for pine tables and pine chairs are reversed.
(d) $\text{OFC}_{\text{Pine Table}}$ increases by $20 while at the same time the $\text{OFC}_{\text{Pine Chair}}$ decreases by $10.
(e) Unit profits for all three types of chairs are increased by $6 each.

4-32 Consider the Classic Furniture product mix problem (Problem 4-28). In answering each of the following questions, be as specific as possible. If it is possible to compute a new profit or production plan, do so.

(a) A part-time employee who works 20 hours per week decided to quit his job. How would this affect the profit and production plan?
(b) Classic has been approached by the factory next door, CabinetsRUs, which has a shortage of both labor and oak. CabinetsRUs proposes to take one full-time employee (who works 30 hours) plus 900 pounds of oak. It has offered $560 as compensation. Should Classic make this trade?
(c) Classic is considering adding a new product, a cherry armoire. The armoire would consume 200 pounds of cherry wood and take 16 hours of labor. Cherry wood costs $9 per pound, and labor costs $12 per hour. The armoire would sell for $2,180. Should this product be made?
(d) What would happen to the solution if a constraint were added to make sure that for every table made, at least two matching chairs were made?

4-33 The Tiger Catering Company is trying to determine the most economical combination of sandwiches to make for a tennis club. The club has asked Tiger to provide 70 sandwiches in a variety to include tuna, tuna and cheese, ham, ham and cheese, and cheese. The club has specified a minimum of 10 each of tuna and ham and 12 each of tuna/cheese and ham/cheese. Tiger makes the sandwiches using the following resources: bread, tuna, ham, cheese, mayonnaise, mustard, lettuce, tomato, packaging material, and labor hours.

The Excel layout and LP Sensitivity Report for Tiger Catering's problem are shown in Screenshots 4-14A and 4-14B, respectively. The objective function coefficients in the screenshots refer to unit cost per item. Each of the following questions is independent of the others.

(a) What is the optimal cost represented by the objective function and what is the optimal sandwich-making plan?
(b) Which constraints are binding?

SCREENSHOT 4-14A
Excel Layout for Tiger Catering Company

	A	B	C	D	E	F	G	H	I
1	**Tiger Catering Company**								
2									
3		Tuna	Tuna/Ch	Ham	Ham/Ch	Cheese			
4	Number to make	10.00	30.00	10.00	12.00	8.00			
5	Cost	$2.42	$2.12	$3.35	$3.02	$2.36	$176.42		
6	**Constraints**								
7	Bread (slices)	2	2	2	2	2	140.00	<=	140
8	Tuna (oz.)	4	3				130.00	<=	130
9	Ham (oz.)			4	3		76.00	<=	100
10	Cheese (oz.)		1		1	4	74.00	<=	80
11	Mayo (oz.)	1.2	0.9	0.5	0.5	0.5	54.00	<=	72
12	Mustard (oz.)			0.2	0.2		4.40	<=	8
13	Lettuce (oz.)	0.25	0.25	0.25	0.25	0.25	17.50	<=	20
14	Tomato (oz.)	0.5	0.5	0.5	0.5	0.5	35.00	<=	40
15	Package (unit)	1	1	1	1	1	70.00	<=	72
16	Labor (hrs)	0.08	0.08	0.08	0.08	0.08	5.60	<=	8
17	Min total	1	1	1	1	1	70.00	>=	70
18	Min Tuna	1					10.00	>=	10
19	Min Tuna/Ch		1				30.00	>=	12
20	Min Ham			1			10.00	>=	10
21	Min Ham/Ch				1		12.00	>=	12
22							**LHS**	**Sign**	**RHS**

(c) What is the range over which the cost for cheese sandwiches could vary without changing the production plan?

(d) What is the range over which the quantity of tuna could vary without changing the combination of binding constraints?

(e) Does this Sensitivity Report indicate the presence of multiple optimal solutions? How do you know?

(f) After the sandwiches are made, how many labor hours remain?

4-34 Consider the Tiger Catering problem (Problem 4-33). For each of the following situations, what would be

SCREENSHOT 4-14B
Solver Sensitivity Report for Tiger Catering Company

Microsoft Excel 14.0 Sensitivity Report
Problems 4-33to37. Tiger Catering Company

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$4	Number to make Tuna	10.00	0.00	2.42	1E+30	0.38
$C$4	Number to make Tuna/Ch	30.00	0.00	2.12	0.24	1E+30
$D$4	Number to make Ham	10.00	0.00	3.35	1E+30	0.99
$E$4	Number to make Ham/Ch	12.00	0.00	3.02	1E+30	0.66
$F$4	Number to make Cheese	8.00	0.00	2.36	0.66	0.24

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$G$7	Bread (slices)	140.00	0.00	140.00	1E+30	0.00
$G$8	Tuna (oz.)	130.00	-0.08	130.00	24.00	6.00
$G$9	Ham (oz.)	76.00	0.00	100.00	1E+30	24.00
$G$10	Cheese (oz.)	74.00	0.00	80.00	1E+30	6.00
$G$11	Mayo (oz.)	54.00	0.00	72.00	1E+30	18.00
$G$12	Mustard (oz.)	4.40	0.00	8.00	1E+30	3.60
$G$13	Lettuce (oz.)	17.50	0.00	20.00	1E+30	2.50
$G$14	Tomato (oz.)	35.00	0.00	40.00	1E+30	5.00
$G$15	Package (unit)	70.00	0.00	72.00	1E+30	2.00
$G$16	Labor (hrs)	5.60	0.00	8.00	1E+30	2.40
$G$17	Min total	70.00	2.36	70.00	0.00	8.00
$G$18	Min Tuna	10.00	0.38	10.00	13.50	10.00
$G$19	Min Tuna/Ch	30.00	0.00	12.00	18.00	1E+30
$G$20	Min Ham	10.00	0.99	10.00	6.00	1.50
$G$21	Min Ham/Ch	12.00	0.66	12.00	8.00	2.00

the impact on the sandwich-making plan and total cost? If it is possible to compute the new cost or sandwich-making plan, do so.
(a) The unit cost for tuna sandwiches decreases by $0.30.
(b) The unit cost for tuna and cheese sandwiches increases to $2.40.
(c) The unit cost for ham sandwiches increases to $3.75.
(d) The unit cost for ham and cheese sandwiches decreases by $0.70.
(e) The club does not want any more than 12 ham sandwiches.
(f) The unit cost for cheese sandwiches decreases to $2.05.

4-35 Consider the Tiger Catering problem (Problem 4-33). For each of the following situations, what would be the impact on the sandwich-making plan and total cost? If it is possible to compute the new cost or sandwich-making plan, do so.
(a) The quantity of tuna available decreases to 120 ounces.
(b) The quantity of ham available increases to 115 ounces.
(c) The quantity of cheese available decreases to 72 ounces.
(d) Tiger is required to deliver a minimum of 13 tuna sandwiches.
(e) Tiger is required to deliver only a minimum of 10 tuna and cheese sandwiches.
(f) Tiger is asked to bring a minimum of only 66 sandwiches.

4-36 Consider the Tiger Catering problem (Problem 4-33). For each of the following situations, what would be the impact on the sandwich-making plan and total cost? If it is possible to compute the new cost or sandwich-making plan, do so.
(a) The cost of ham sandwiches and the cost of ham and cheese sandwiches each decreases by $0.35.
(b) The cost of both ham and cheese sandwiches and cheese sandwiches increases by $0.60.
(c) The cost of tuna decreases by $0.10 per ounce. (*Hint:* Note that tuna sandwiches use 4 ounces of tuna and tuna/cheese sandwiches use 3 ounces of tuna.)
(d) The availability of tuna increases by 10 ounces and the availability of ham decreases by 10 ounces.
(e) A 16-oz jar of mustard is sent by mistake instead of a 16-oz jar of mayonnaise. (*Hint:* This would decrease the quantity of mayonnaise by 16 ounces and increase the quantity of mustard by 16 ounces.)

4-37 Consider the Tiger Catering problem (Problem 4-33). In answering each of the following questions, be as specific as possible. If it is possible to compute a new cost or sandwich-making plan, do so.
(a) An additional pound of tuna can be obtained for a premium of $1.50. Should this tuna be purchased?
(b) The tennis club is willing to accept fewer ham and ham and cheese sandwiches. How many of these sandwiches would Tiger try to substitute with other types before they would not be able to predict their new total cost?
(c) The tennis club wants to include a dill pickle slice with each meat sandwich order. If Tiger finds an average of 18 slices in a 2-pound pickle jar, how many jars should be included with the club's order?

Case Study

Coastal States Chemicals and Fertilizers

In December 2005, Bill Stock, general manager for the Louisiana Division of Coastal States Chemicals and Fertilizers, received a letter from Fred McNair of the Cajan Pipeline Company, which notified Coastal States that priorities had been established for the allocation of natural gas. The letter stated that Cajan Pipeline, the primary supplier of natural gas to Coastal States, might be instructed to curtail natural gas supplies to its industrial and commercial customers by as much as 40% during the ensuing winter months. Moreover, Cajan Pipeline had the approval of the Federal Power Commission (FPC) to curtail such supplies.

Possible curtailment was attributed to the priorities established for the use of natural gas:

First priority:	residential and commercial heating
Second priority:	commercial and industrial users that use natural gas as a source of raw material
Third priority:	commercial and industrial users whereby natural gas is used as boiler fuel

Almost all of Coastal States' uses of natural gas were in the second and third priorities. Hence, its plants were certainly subject to brownouts, or natural gas curtailments. The occurrence and severity of the brownouts depended on a number of complex factors. First, Cajan Pipeline was part of an interstate transmission network that delivered natural gas to residential and commercial buildings on the Atlantic coast and in Northeastern regions of the United States. Hence, the severity of the forthcoming winter in these regions would have a direct impact on the use of natural gas.

Second, the demand for natural gas was soaring because it was the cleanest and most efficient fuel. There were almost no environmental problems in burning natural gas. Moreover, maintenance problems due to fuel-fouling in fireboxes and boilers were negligible with natural gas systems. Also, burners were much easier to operate with natural gas than with oil or coal.

Finally, the supply of natural gas was dwindling. The traditionally depressed price of natural gas had discouraged new exploration for gas wells; hence, shortages appeared imminent.

Stock and his staff at Coastal States had been aware of the possibility of shortages of natural gas and had been investigating ways of converting to fuel oil or coal as a substitute for natural gas. Their plans, however, were still in the developmental stages. Coastal States required an immediate contingency plan to minimize the effect of a natural gas curtailment on its multiplant, operations. The obvious question was, what operations should be curtailed, and to what extent could the adverse effect upon profits be minimized? Coastal States had approval from the FPC and Cajan Pipeline to specify which of its plants would bear the burden of the curtailment if such cutbacks were necessary. McNair, of Cajan Pipeline, replied, "It's your 'pie': we don't care how you divide it if we make it smaller."

The Model

Six plants of Coastal States Louisiana Division were to share in the "pie." They were all located in the massive Baton Rouge–Geismar–Gramercy industrial complex along the Mississippi River between Baton Rouge and New Orleans. Products manufactured at those plants that required significant amounts of natural gas were phosphoric acid, urea, ammonium phosphate, ammonium nitrate, chlorine, caustic soda, vinyl chloride monomer, and hydrofluoric acid.

Stock called a meeting of members of his technical staff to discuss a contingency plan for allocation of natural gas among the products if a curtailment developed. The objective was to minimize the impact on profits. After detailed discussion, the meeting was adjourned. Two weeks later, the meeting reconvened. At this session, the data in Table 4.3 were presented.

Coastal States' contract with Cajan Pipeline specified a maximum natural gas consumption of 36,000,000 cubic feet per day for all six member plants. With these data, the technical staff proceeded to develop a model that would specify changes in production rates in response to a natural gas curtailment. (Curtailments are based on contracted consumption and not current consumption.)

Discussion Questions

1. Develop a contingency model and specify the production rates for each product for
   (a) a 20% natural gas curtailment.
   (b) a 40% natural gas curtailment.
2. What impact will the natural gas shortage have on company profits?
3. Develop the Sensitivity Report for the 20% natural gas curtailment model. Use this report to answer the following questions. Each question is independent of the others.
   (a) Interpret the shadow prices for the natural gas availability constraint and for the two constraints that limit the maximum phosphoric acid and chlorine that Coastal can produce.
   (b) Brenda Lamb, Bill Stock's marketing manager, believes that due to increased competition she may have to decrease the unit profit contributions for all products by 3.5% each. What is the impact of this decrease on the production values? On the total profit?
   (c) Jose Fernandez, Bill Stock's production manager, thinks that he can increase the maximum production rate for chlorine and vinyl chloride monomer to 80% of capacity. For all other products, he thinks he can increase the maximum production rate to 100% of capacity. What would be the impact of this change on the total profit?

TABLE 4.3 Contribution to Profit and Overhead

PRODUCT	CONTRIBUTION ($ PER TON)	CAPACITY (TONS PER DAY)	MAXIMUM PRODUCTION RATE (PERCENTAGE OF CAPACITY)	NATURAL GAS CONSUMPTION (1,000 CU. FT. PER TON)
Phosphoric acid	60	400	80	5.5
Urea	80	250	80	7.0
Ammonium phosphate	90	300	90	8.0
Ammonium nitrate	100	300	100	10.0
Chlorine	50	800	60	15.0
Caustic soda	50	1,000	60	16.0
Vinyl chloride monomer	65	500	60	12.0
Hydrofluoric acid	70	400	80	11.0

(d) Bill Stock thinks he can persuade Coastal's Mississippi Division to give him 1,000,000 cubic feet of its allotment of natural gas from Cajan Pipeline. However, due to the Mississippi Division's pricing contract with Cajan Pipeline, this additional amount of natural gas will cost Stock an additional \$1.50 per 1,000 cubic feet (over current costs). Should Stock pursue this option? If so, what is the impact of this additional gas on his total profit? What is the impact if Bill Stock can persuade the Mississippi Division to give him 3,000,000 cubic feet of its allotment of natural gas from Cajan Pipeline?

4. Redo question 3 using the Sensitivity Report for the 40% natural gas curtailment model. In addition, interpret the reduced cost for caustic soda.

Source: Jerry Kinard, Western Carolina University, and Brian Kinard, University of North Carolina - Wilmington.

CHAPTER 5

Transportation, Assignment, and Network Models

LEARNING OBJECTIVES

After completing this chapter, students will be able to:

1. Structure special LP network flow models.
2. Set up and solve transportation models, using Excel's Solver.
3. Set up and solve transportation models with Max-Min and Min-Max objectives.
4. Extend the basic transportation model to include transshipment points.
5. Set up and solve maximal-flow network models, using Excel's Solver.
6. Set up and solve shortest-path network models, using Excel's Solver.
7. Connect all points of a network while minimizing total distance, using the minimal-spanning tree model.

CHAPTER OUTLINE

5.1 Introduction

In this chapter, we examine six different examples of special linear programming (LP) models, called *network flow models*: (1) transportation, (2) transshipment, (3) assignment, (4) maximal-flow, (5) shortest-path, and (6) minimal-spanning tree models. Networks consist of nodes (or points) and arcs (or lines) that connect the nodes together. Roadways, telephone systems, and citywide water systems are all examples of networks.

Transportation Model

Transportation models ***deal with distribution of goods from supply points to demand points at minimum cost.***

The **transportation model** deals with the distribution of goods from several supply points (also called **origins**, or **sources**) to a number of demand points (also called **destinations**, or **sinks**). Usually, we have a given capacity of goods at each source and a given requirement for the goods at each destination. The most common objective of a transportation model is to schedule shipments from sources to destinations so that total production and transportation costs are minimized. Occasionally, transportation models can have a maximization objective (e.g., maximize total profit of shipping goods from sources to destinations).

Transportation models can also be used when a firm is trying to decide where to locate a new facility. Before opening a new warehouse, factory, or office, it is good practice to consider a number of alternative sites. Good financial decisions concerning facility location also involve minimizing total production and transportation costs for the entire system.

Transshipment Model

In **transshipment models*****, some points can have shipments that arrive as well as leave.***

In a basic transportation model, shipments either leave a supply point or arrive at a demand point. An extension of the transportation model is called the **transshipment model**, in which a point can have shipments that both arrive and leave. An example would be a warehouse where shipments arrive from factories and then leave for retail outlets. It may be possible for a firm to achieve cost savings (economies of scale) by consolidating shipments from several factories at the warehouse and then sending them together to retail outlets. This type of approach is the basis for the *hub-and-spoke* system of transportation employed by most major U.S. airlines. For example, most travel on Delta Air Lines from the Western U.S. to the Eastern U.S. (or vice versa) involves a connection through Delta's hub in Atlanta, Georgia.

Assignment Model

An **assignment model** ***seeks to find the optimal one-to-one assignment of people to projects, jobs to machines, and so on.***

The **assignment model** refers to the class of LP problems that involve determining the most efficient assignment of people to projects, salespeople to territories, contracts to bidders, jobs to machines, and so on. The typical objective is to minimize total cost or total time of performing the tasks at hand, although a maximization objective is also possible. An important characteristic of assignment models is that each job or worker can be assigned to at most one machine or project, and vice versa.

Maximal-Flow Model

Consider a network that has a specific starting point (called the *origin*) and a specific ending point (called the *destination*). The arcs in the network have capacities that limit the amounts of flow that can occur on them. These capacities can be different for different arcs. The **maximal-flow model** finds the maximum flow that can occur from the origin to the destination through this network. This model can be used to determine, for example, the maximum number of vehicles (cars, trucks, and so forth) that can go through a network of roads from one location to another.

A **maximal-flow model** ***finds the maximum flow possible through a network.***

Shortest-Path Model

Consider a network that has a specified origin and a specified destination. The arcs in the network are such that there are many paths available to go from the origin to the destination. The **shortest-path model** finds the shortest path or route through this network from the origin to the destination. For example, this model can be used to find the shortest distance and route from one city to another through a network of roads. The *length* of each arc can be a function of its distance, travel time, travel cost, or any other measure.

A **shortest-path model** ***finds the shortest route from an origin to a destination.***

Minimal-Spanning Tree Model

A **minimal-spanning tree model** *connects all nodes in a network while minimizing total distance.*

The **minimal-spanning tree model** determines the path through the network that connects all the points. The most common objective is to minimize total distance of all arcs used in the path. For example, when the points represent houses in a subdivision, the minimal-spanning tree model can be used to determine the best way to connect all the houses to electrical power, water systems, and so on, in a way that minimizes the total distance or length of power lines or water pipes.

All the examples used to describe the various network models in this chapter are rather small (compared to real problems), to make it easier for you to understand the models. In some cases, the small size of these network examples may make them solvable by inspection or intuition. For larger real-world problems, however, finding a solution can be very difficult and requires the use of computer-based modeling approaches, as discussed here.

5.2 Characteristics of Network Models

A **node** *is a specific point or location in a network.*

An **arc** *connects two nodes to each other.*

Each of the circles (numbered 1 to 5) in Figure 5.1 is called a *node*. A node can be defined as the location of a specific point on the network. For example, nodes could represent cities on a road network, Ethernet ports on a campus computer network, houses in a city's water supply network, etc. An *arc* is a line that connects two nodes to each other. Arcs could represent roads that connect cities, computer network cables between various Ethernet ports, pipes that carry water to each house, etc. Figure 5.1 shows a network that has 5 nodes and 10 arcs.

Types of Arcs

Arcs can be one way or two way.

As shown in Figure 5.1, it is not necessary for an arc to exist between every pair of nodes in a network. A network that does have arcs between all pairs of nodes is called a *fully connected* network. Arcs can be either *unidirectional* (meaning that flow can occur only in one direction, as in a one-way road) or *bidirectional* (meaning that flow can occur in either direction). From a modeling perspective, it is convenient to represent a bidirectional arc with a pair of unidirectional arcs with opposite flow directions. This concept is illustrated in Figure 5.1 by the pairs of arcs between nodes 1 and 3, nodes 3 and 5, and nodes 2 and 4. Flows between all other pairs of nodes in Figure 5.1 are unidirectional.

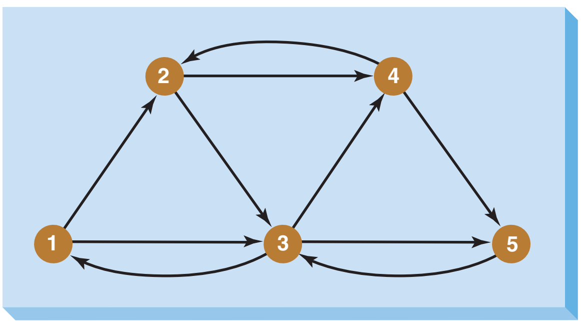
Arcs can also be classified as capacitated or uncapacitated. A *capacitated* arc has a limited capacity, as in the case of a water pipe or a road. An *uncapacitated* arc, in contrast, can support an unlimited flow. In practice, this does not necessarily mean that the arc has infinite capacity. Rather, it means that the arc's capacity is so high that it is not a constraint in the model. An example could be a road in a small, rural area. The road rarely encounters traffic congestion because its capacity is far greater than the number of vehicles traveling on it at any one time.

Types of Nodes

Nodes can be **origins, destinations,** *or* **transshipment** *nodes.*

Nodes can be classified as *supply nodes*, *demand nodes*, or *transshipment* nodes. A supply node, also known as an origin or a source, denotes a location such as a factory that creates goods. That is, goods enter the network at that node.

FIGURE 5.1
Example of a Network
Note: Nodes are circles; arcs are lines.



A demand node, also known as a destination or a sink, denotes a location such as a retail outlet that consumes goods. That is, goods leave the network at that node.

A transshipment node denotes a location through which goods pass on their way to or from other locations. In many practical networks, the same node can be a combination of a supply node, a demand node, and a transshipment node. For example, in the case of Delta Air Lines, Atlanta is a supply node for people starting their trip from Atlanta, a demand node for people ending their trip in Atlanta, and a transshipment node for people taking connecting flights through Atlanta.

Why are transportation models (and other network flow models) a special case of LP models? The reason is that many network models share some common characteristics, as follows:

1. In *all* network models, the decision variables represent the amounts of flows (or shipments) that occur on the unidirectional arcs in the network. For example, the LP model for the network shown in Figure 5.1 will have 10 decision variables representing the amounts of flows on the 10 unidirectional arcs.
2. Second, there will be a *flow balance* constraint written for each node in the network. These flow balance constraints calculate the *net flow* at each node (i.e., the difference between the total flow on all arcs entering a node and the total flow on all arcs leaving the node):

$$\text{Net flow} = (\text{total flow } in \text{ to node}) - (\text{Total flow } out \text{ of node}) \quad \textbf{(5-1)}$$

The net flow at a node is the difference between the total flow in to the node and the total flow out of the node.

At supply nodes, the total flow *out* of the node will exceed the total flow *in* to the node because goods are created at the node. In fact, at a pure supply node, there will only be flows out of the node (i.e., the total flow out will be a positive quantity) and no flows in to the node (i.e., the total flow in will be zero). The net flow represents the amount of goods created (i.e., the supply) at that node. Note that because the flow out is larger than the flow in at supply nodes, the resulting net flow will be a *negative* quantity. For this reason, as we will see shortly, we will express supply values (i.e., net flows at supply nodes) as negative numbers in our model.

On the other hand, at demand nodes, the total flow *out* of the node will be less than the total flow *in* to the node because goods are consumed at the node. At a pure demand node, there will only be flows in to the node (i.e., the total flow in will be a positive quantity) and no flows out of the node (i.e., the total flow out will be zero). The net flow, which represents the amount of goods consumed (i.e., the demand) at that node, will therefore be a *positive* quantity. For this reason, as we will again see shortly, we will express demand values (i.e., net flows at demand nodes) as positive numbers in our model.[1]

At pure transshipment nodes, goods are neither created nor consumed. The total flow *out* of such a node equals the total flow *in* to the node, and the net flow is therefore zero.
3. The constraint coefficients (i.e., the coefficients in front of decision variables in a constraint) for all flow balance constraints and most other problem-specific constraints in network models equal either 0 or 1. That is, if a decision variable exists in a constraint in a network model, its constraint coefficient is usually 1. This special trait allows network flow models to be solved very quickly, using specialized algorithms. However, we use Solver in the same manner as in Chapters 2 and 3 to solve these models here.

If all supplies and demands are integers, all flows in a network will also be integer values.

4. If all supply values at the supply nodes and all demand values at the demand nodes are whole numbers (i.e., integer values), the solution to a network model will automatically result in integer values for the decision variables, even if we don't impose an explicit condition to this effect. This property is especially useful in modeling the assignment and shortest-path models discussed later in this chapter.

[1] Some students may find it confusing, and somewhat counterintuitive, to write supply values as negative numbers. To avoid this, you may prefer to define net flows differently at supply nodes and demand nodes, as follows:

$$\text{Net flow at demand nodes} = (\text{Total flow } in \text{ to node}) - (\text{Total flow } out \text{ of node})$$
$$\text{Net flow at supply nodes} = (\text{Total flow } out \text{ of node}) - (\text{Total flow } in \text{ to node})$$

In this textbook, however, we have chosen to use a single definition of net flow at both these types of nodes. We will therefore express supply values as negative numbers and demand values as positive numbers in our models.

IN ACTION Improving Freight Car Assignment at Union Pacific Railroad

Union Pacific Railroad (UP), the largest railroad in North America, has about 32,300 miles of track, 8,500 locomotives, 104,700 freight cars, 50,000 employees, and an annual payroll of $3.7 billion. UP faces a difficult problem in assigning empty freight cars to customers since these assignments depend on many factors including the location of empty cars, the demand urgency, and whether or not cars can be substituted. Working with researchers at Purdue University, UP developed a transportation optimization model to address this problem, with the objective of reducing transportation costs while improving delivery time and customer satisfaction. The resulting real-time decision system is compatible with UP's operational practices, and can be used to assign and reassign empty freight cars in a short time.

UP has achieved significant savings in transportation costs due to the implementation of this system. In addition, UP experienced a 35% ROI due to reductions in demand fulfillment personnel that was made possible by this project. As an added benefit, this project also helped UP obtain a better understanding of the importance and the accuracy required for many of the data elements, and gain valuable insight into some of the issues and tradeoff that arise in automating the assignment of empty freight cars.

Source: Based on A. K. Narisetty et al. "An Optimization Model for Empty Freight Car Assignment at Union Pacific Railroad," *Interfaces* 38, 2 (March–April 2008): 89–102.

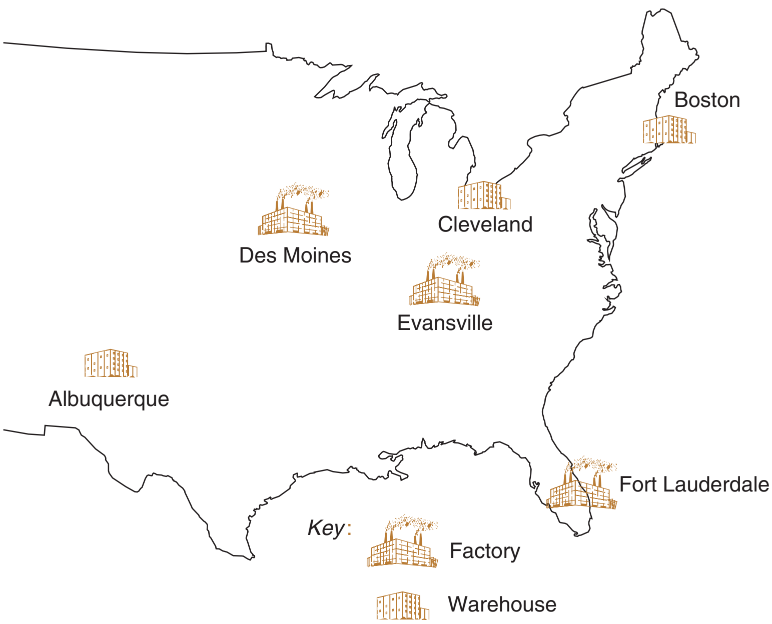
5.3 Transportation Model

Let us begin to illustrate the transportation model with an example dealing with the Executive Furniture Company. This company manufactures office desks at three locations: Des Moines, Evansville, and Fort Lauderdale. The firm distributes the desks through regional warehouses located in Albuquerque, Boston, and Cleveland (see Figure 5.2). Estimates of the monthly supplies available at each factory and the monthly desk demands at each of the three warehouses are shown in Figure 5.3.

Our goal is to select the shipping routes and units to be shipped to minimize total transportation cost.

The firm has found that production costs per desk are identical at each factory, and hence the only relevant costs are those of shipping from each factory to each warehouse. These costs, shown in Table 5.1, are assumed to be constant, regardless of the volume shipped.[2] The transportation problem can now be described as *determining the number of desks to be shipped on each route so as to minimize total transportation cost*. This, of course, must be done while observing the restrictions regarding factory supplies and warehouse demands.

FIGURE 5.2
Geographic Locations of Executive Furniture's Factories and Warehouses



[2] The other assumptions that held for LP problems (see Chapter 2) are still applicable to transportation problems.

FIGURE 5.3
Network Model for Executive Furniture—Transportation

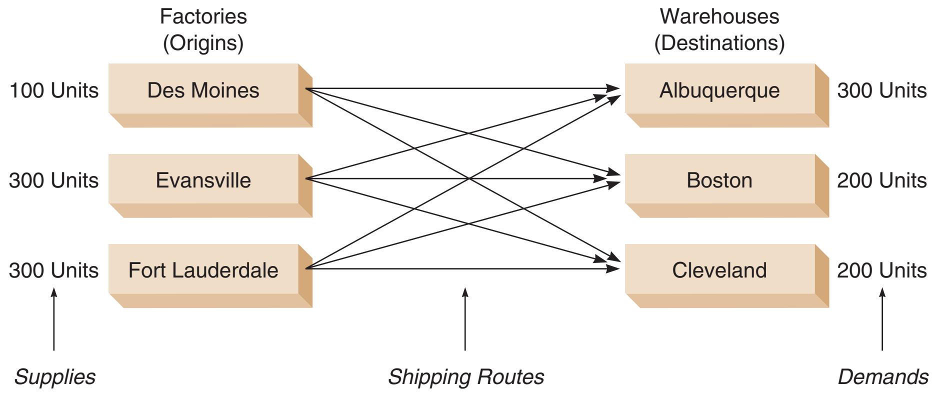


TABLE 5.1
Transportation Costs per Desk for Executive Furniture

	TO		
FROM	ALBUQUERQUE	BOSTON	CLEVELAND
Des Moines	\$5	\$4	\$3
Evansville	\$8	\$4	\$3
Fort Lauderdale	\$9	\$7	\$5

Balanced supply and demand occurs when total supply equals total demand.

We see in Figure 5.3 that the total factory supply available (700) is exactly equal to the total warehouse demand (700). When this situation of equal total demand and total supply occurs (something that is rather unusual in real life), a **balanced model** is said to exist. Later in this section we look at how to deal with *unbalanced* models—namely, those in which total demands are greater than or less than total supplies.

LP Formulation for Executive Furniture's Transportation Model

Because there are three factories (Des Moines, Evansville, and Fort Lauderdale) and three warehouses (Albuquerque, Boston, and Cleveland), there are nine potential shipping routes. We therefore need nine decision variables to define the number of units that would be shipped from each supply node (factory) to each demand node (warehouse). In general, the number of decision variables in a basic transportation model is the number of supply nodes multiplied by the number of demand nodes.

It is convenient to express all network flows by using double-subscripted variables.

Recall from section 5.2 that in the transportation model (as well as in other network flow models), decision variables denote the flow between two nodes in the network. Therefore, it is convenient to represent these flows by using double-subscripted decision variables. We let the first subscript represent the supply node and the second subscript represent the demand node of the flow. Hence, for the Executive Furniture example, let

$$X_{ij} = \text{number of desks shipped from factory } i \text{ to warehouse } j$$

where

$$i = \text{D (for Des Moines), E (for Evansville), or F (for Fort Lauderdale)}$$
$$j = \text{A (for Albuquerque), B (for Boston), or C (for Cleveland)}$$

OBJECTIVE FUNCTION The objective function for this model seeks to minimize the total transportation cost and can be expressed as

$$\text{Minimize total shipping costs} = 5X_{DA} + 4X_{DB} + 3X_{DC} + 8X_{EA} + 4X_{EB} + 3X_{EC} + 9X_{FA} + 7X_{FB} + 5X_{FC}$$

We write a **flow balance** *constraint for each node in the network.*

CONSTRAINTS As discussed earlier, we need to write *flow balance* constraints for each node in the network. Because the Executive Furniture example is a balanced model, we know that all desks will be shipped from the factories and all demand will be satisfied at the warehouses. The number of desks shipped from each factory will therefore be equal to the number of desks available, and the number of desks received at each warehouse will be equal to the number of desks required.

We write a supply constraint *for each factory.*

SUPPLY CONSTRAINTS The *supply constraints* deal with the supplies available at the three factories. At all factories, the total flow *in* is zero because there are no arcs coming into these nodes. The net flow at the Des Moines factory (for example) can therefore be expressed as

$$\begin{aligned}\text{Net flow at Des Moinse} &= (\text{Total flow } in \text{ to Des Moines}) - (\text{Total flow } out \text{ of Des Moines})\\ &= (0) - (X_{DA} + X_{DB} + X_{DC})\end{aligned}$$

Supplies are usually written as negative quantities.

This net flow is equal to the total number of desks available (supply) at Des Moines. Recall from section 5.2 that we express supply values as negative numbers in our network flow balance constraints. Therefore, the right-hand side (RHS) is written as −100 in this equation:

$$\text{Net flow at Des Moines} = -X_{DA} - X_{DB} - X_{DC} = -100 \qquad \text{(Des Moines supply)}$$

Likewise, the flow balance constraints at the other factories can be expressed as

$$-X_{EA} - X_{EB} - X_{EC} = -300 \qquad \text{(Evansville supply)}$$
$$-X_{FA} - X_{FB} - X_{FC} = -300 \qquad \text{(Fort Lauderdale supply)}$$

As noted earlier, if you prefer not to have negative quantities on the RHS of the supply constraints, you can redefine the net flow at *supply nodes* as equal to (Total flow *out* of node – Total flow *in* to node), and rewrite these equations as

$$X_{DA} + X_{DB} + X_{DC} = 100 \qquad \text{(Des Moines supply)}$$
$$X_{EA} + X_{EB} + X_{EC} = 300 \qquad \text{(Evansville supply)}$$
$$X_{FA} + X_{FB} + X_{FC} = 300 \qquad \text{(Fort Lauderdale supply)}$$

We write a demand constraint for each warehouse.

DEMAND CONSTRAINTS Now, let us model the *demand constraints* that deal with the warehouse demands. At all warehouses, the total flow *out* is zero because there are no arcs leaving from these nodes. The net flow at the Albuquerque warehouse, for example, can therefore be expressed as

$$\begin{aligned}\text{Net flow at Albuquerque} &= (\text{Total flow } in \text{ to Albuquerque}) - (\text{Total flow } out \text{ of Albuquerque})\\ &= (X_{DA} + X_{EA} + X_{FA}) - (0)\end{aligned}$$

Net flow at a demand node is written as a positive number.

This net flow is equal to the total number of desks required (demand) at Albuquerque. Recall from section 5.2 that we express demand values at demand nodes as positive numbers in our network flow balance constraints. Therefore,

$$\text{Net flow at Albuquerque} = (X_{DA} + X_{EA} + X_{FA}) = 300 \qquad \text{(Albuquerque demand)}$$

Likewise, the flow balance constraints at the other warehouses can be expressed as

$$X_{DB} + X_{EB} + X_{FB} = 200 \qquad \text{(Boston demand)}$$
$$X_{DC} + X_{EC} + X_{FC} = 200 \qquad \text{(Cleveland demand)}$$

In general, the number of constraints in the basic transportation model is the sum of the number of supply nodes and the number of demand nodes. There could, however, be other problem-specific constraints that restrict shipments in individual routes. For example, if we wished to ensure that no more than 100 desks are shipped from Evansville to Cleveland, an additional constraint in the model would be $X_{EC} \leq 100$.

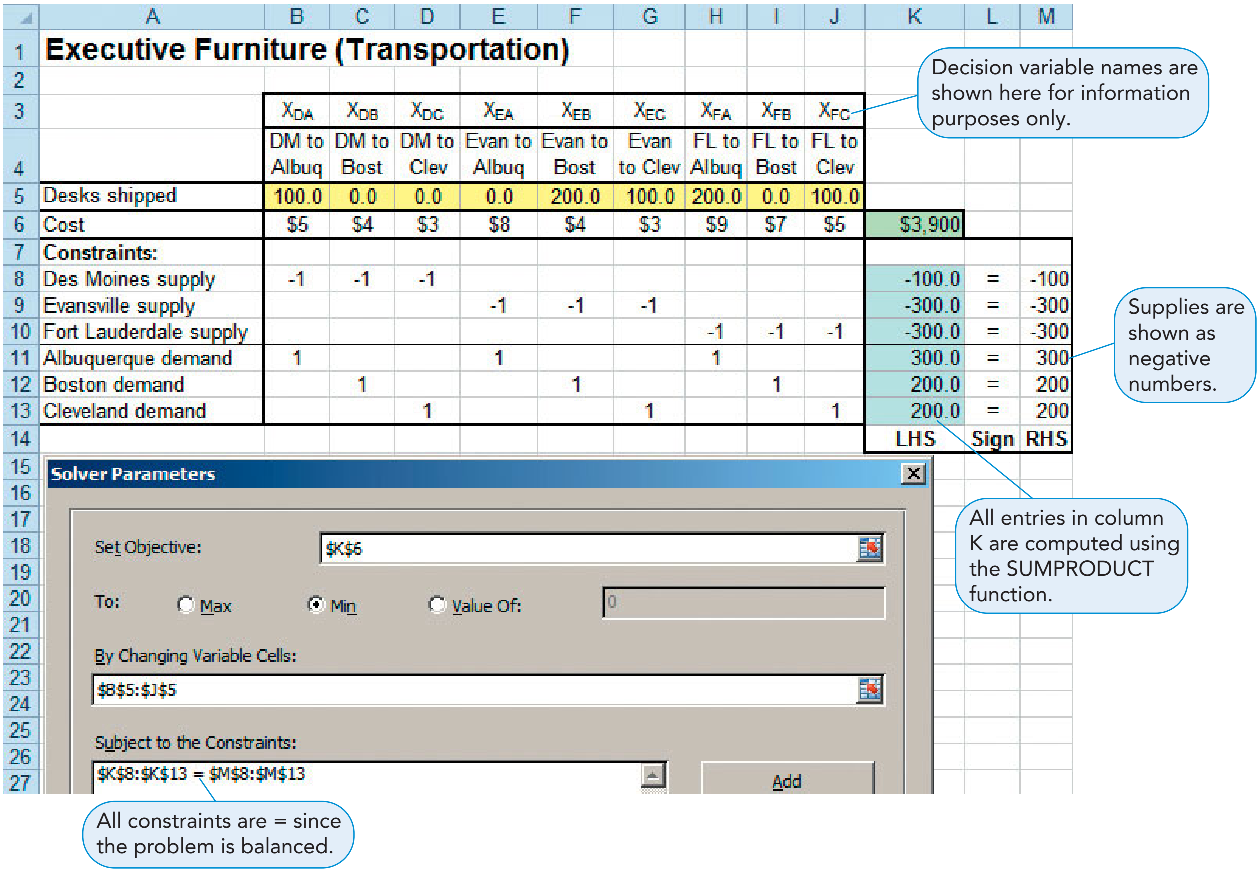
Solving the Transportation Model Using Excel

WWW

File: 5-1.xls

Screenshot 5-1 shows the Excel setup and Solver entries for Executive Furniture's transportation model. Consistent with our choice of using the same definition of net flow at all types of nodes, all supply values are expressed here as negative values (cells M8:M10). The Excel layout in Screenshot 5-1 follows the same logic as in Chapter 3. This means that (1) each decision variable is modeled in a separate column of the worksheet, and (2) the objective function and left-hand side (LHS) formulas for all constraints are computed using Excel's SUMPRODUCT function.

SCREENSHOT 5-1 **Excel Layout and Solver Entries for Executive Furniture—Transportation**



Excel Notes

- The Companion Website for this textbook, at www.pearsonhighered.com/balakrishnan, contains the Excel file for each sample problem discussed here. The relevant file name is shown on the margin next to each example.
- In each of our Excel layouts, for clarity, changing variable cells are shaded yellow, the objective cell is shaded green, and cells containing the LHS formula for each constraint are shaded blue.

The optimum solution for Executive Furniture Company is to ship 100 desks from Des Moines to Albuquerque, 200 desks from Evansville to Boston, 100 desks from Evansville to Cleveland, 200 desks from Fort Lauderdale to Albuquerque, and 100 desks from Fort Lauderdale to Cleveland. The total shipping cost is $3,900. Observe that because all supplies and demands were integer values, all shipments turned out to be integer values as well.

Alternate Excel Layout for the Transportation Model

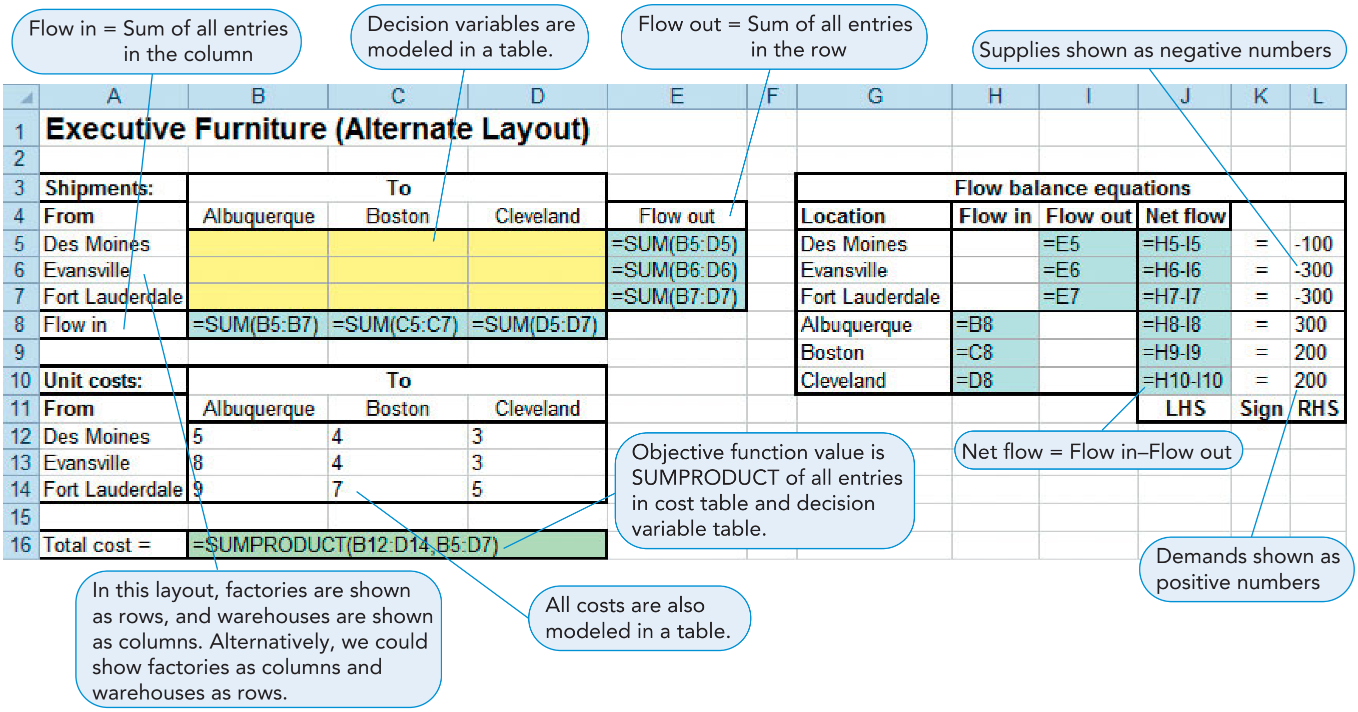
The alternate Excel layout for network flow models uses a tabular form to model the flows.

File: 5-2.xls

For many network models, the number of arcs (and, hence, decision variables) could be quite large. Modeling the problem by using the layout used in Screenshot 5-1 can therefore become quite cumbersome. For this reason, it may be more convenient to model network flow models in Excel in such a way that decision variables are in a *tabular* form, with rows (for example) denoting supply nodes and columns denoting demand nodes. The formula view of the alternate Excel layout for Executive Furniture's transportation model is shown in Screenshot 5-2A, and the optimal solution is shown in Screenshot 5-2B.

By adding the row (or column) entries, we can easily calculate the appropriate total flows *out* and total flows *in* at each node. It is then possible to model a flow balance constraint for each node by calculating each net flow as the difference between the total flow *in* to the node and the

SCREENSHOT 5-2A **Formula View of Alternate Excel Layout for Executive Furniture—Transportation**

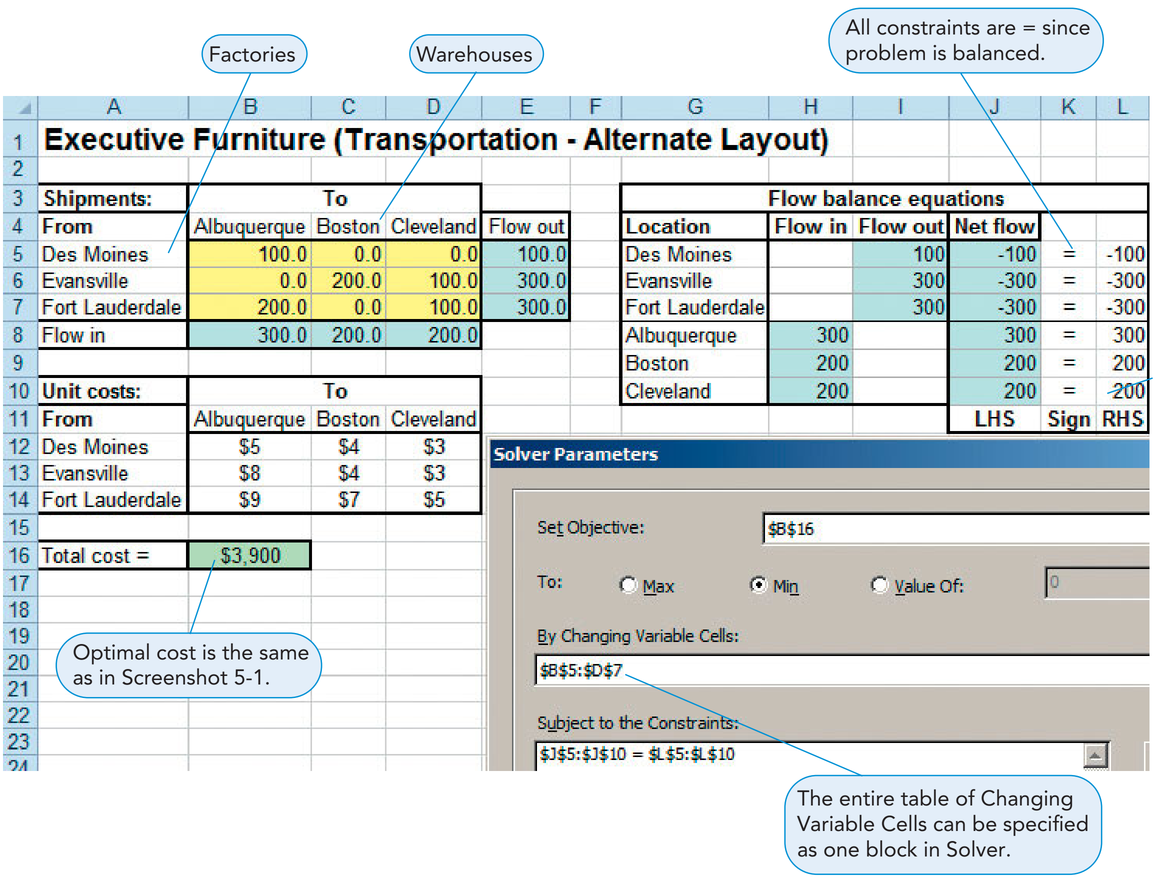


Executive Furniture (Alternate Layout)

	A	B	C	D	E
3	**Shipments:**		To		
4	**From**	Albuquerque	Boston	Cleveland	Flow out
5	Des Moines				=SUM(B5:D5)
6	Evansville				=SUM(B6:D6)
7	Fort Lauderdale				=SUM(B7:D7)
8	Flow in	=SUM(B5:B7)	=SUM(C5:C7)	=SUM(D5:D7)	
10	**Unit costs:**		To		
11	**From**	Albuquerque	Boston	Cleveland	
12	Des Moines	5	4	3	
13	Evansville	8	4	3	
14	Fort Lauderdale	9	7	5	
16	Total cost =	=SUMPRODUCT(B12:D14,B5:D7)			

	G	H	I	J	K	L
3	**Flow balance equations**					
4	**Location**	**Flow in**	**Flow out**	**Net flow**		
5	Des Moines		=E5	=H5-I5	=	-100
6	Evansville		=E6	=H6-I6	=	-300
7	Fort Lauderdale		=E7	=H7-I7	=	-300
8	Albuquerque	=B8		=H8-I8	=	300
9	Boston	=C8		=H9-I9	=	200
10	Cleveland	=D8		=H10-I10	=	200
11				**LHS**	**Sign**	**RHS**

SCREENSHOT 5-2B **Solver Entries for Alternate Layout of Executive Furniture—Transportation**



Executive Furniture (Transportation - Alternate Layout)

	A	B	C	D	E
3	**Shipments:**		To		
4	**From**	Albuquerque	Boston	Cleveland	Flow out
5	Des Moines	100.0	0.0	0.0	100.0
6	Evansville	0.0	200.0	100.0	300.0
7	Fort Lauderdale	200.0	0.0	100.0	300.0
8	Flow in	300.0	200.0	200.0	
10	**Unit costs:**		To		
11	**From**	Albuquerque	Boston	Cleveland	
12	Des Moines	$5	$4	$3	
13	Evansville	$8	$4	$3	
14	Fort Lauderdale	$9	$7	$5	
16	Total cost =	$3,900			

	G	H	I	J	K	L
3	**Flow balance equations**					
4	**Location**	**Flow in**	**Flow out**	**Net flow**		
5	Des Moines		100	-100	=	-100
6	Evansville		300	-300	=	-300
7	Fort Lauderdale		300	-300	=	-300
8	Albuquerque	300		300	=	300
9	Boston	200		200	=	200
10	Cleveland	200		200	=	200
11				**LHS**	**Sign**	**RHS**

total flow *out* of the node. To make this easy to understand, we have arranged these entries in a separate box in our Excel layout. For example, cells H5, I5, and J5 show, respectively, the flow *in*, flow *out*, and net flow at Des Moines. There is no flow *in* (i.e., it is equal to zero), while the flow *out* is the sum of cells B5:D5 (as computed in cell E5). The net flow is the difference between cells H5 and I5 (i.e., H5 – I5). The RHS of this net flow constraint is expressed in the model as −100 (in cell L5) because the supply at Des Moines is 100. Likewise, the net flow at Albuquerque (cell J8) is computed as (H8 – I8), where cell H8 (flow *in*) is the sum of cells B5:B7, and cell I8 (flow *out*) is zero. The RHS of this net flow constraint is expressed as +300 because the demand at Albuquerque is 300. Since this is a balanced transportation model, all supply and all demand constraints have = signs in the model.

Note that the optimal solution resulting from this alternate layout, shown in Screenshot 5-2B, is the same as the one shown in Screenshot 5-1.

Unbalanced Transportation Models

A transportation model is **unbalanced** *if the total supply does not equal the total demand.*

In the Executive Furniture example, the total supply from the three factories equals the total requirements at the three warehouses. All supply and demand constraints could therefore be specified as equalities (i.e., using the = sign). But what if the total supply exceeds the total demand, or vice versa? In these cases, we have an **unbalanced model**, and the supply or demand constraints need to be modified accordingly.

There are two possible scenarios: (1) Total supply exceeds the total demand and (2) total supply is less than the total demand.

TOTAL SUPPLY EXCEEDS THE TOTAL DEMAND If total supply exceeds total demand, all demands will be fully satisfied at the demand nodes, but some of the supplies at one or more supply nodes will not need to be shipped out. That is, they will remain at the supply nodes. To allow for this possibility, the total flow *out* of each supply node should be permitted to be smaller than the supply at that node. The total flow *in* to the demand nodes will, however, continue to be written with = signs.

If total supply exceeds total demand, the supply constraints are written as inequalities.

Assume that the supply and demand values in the Executive Furniture example are altered so that the total supply at the three factories exceeds the total demand at the three warehouses. For example, assume that the monthly supply at Des Moines is 150 desks. The total supply is now 750 desks, while the total demand is only 700 desks. The total flow out of Des Moines (i.e., $X_{DA} + X_{DB} + X_{DC}$) should now be permitted to be smaller than the total supply. That is, the constraint needs to be written as an inequality, as follows:

$$X_{DA} + X_{DB} + X_{DC} \leq 150$$

In keeping with our convention of writing net flow constraints in which flows *out* of nodes have negative constraint coefficients and the supply values at supply nodes are expressed as negative numbers, we multiply this expression through by −1 and rewrite the supply constraint for Des Moines as

$$-X_{DA} - X_{DB} - X_{DC} \geq -150 \qquad \text{(Des Moines supply)}$$

Note that when we multiply the equation by −1, the sign changes from ≤ to ≥. Likewise, the supply constraints at the Evansville and Fort Lauderdale factories would need to be revised as

$$-X_{EA} - X_{EB} - X_{EC} \geq -300 \qquad \text{(Evansville supply)}$$
$$-X_{FA} - X_{FB} - X_{FC} \geq -300 \qquad \text{(Fort Lauderdale supply)}$$

Because the demand constraints at the three warehouses will continue to be written as = constraints, the solution will show movements of 700 desks between the factories and the warehouses. The remaining supply of 50 desks will remain at their original locations at one or more of the factories.

TOTAL SUPPLY IS LESS THAN THE TOTAL DEMAND When total supply is less than total demand, all items at the supply nodes will be shipped out, but demands at one or more demand nodes will remain unsatisfied. To allow for this possibility, the total flow *in* at demand nodes should be permitted to be smaller than the requirement at those nodes. The total flow *out* of supply nodes will, however, continue to be written with = signs.

If total demand exceeds total supply, the demand constraints are written as inequalities.

Assume that the supply and demand values in the Executive Furniture example are altered so that the total supply at the three factories is now *less* than the total demand at the three warehouses. For example, assume that the monthly demand at the Albuquerque warehouse is 350 desks. The total flow *in* to Albuquerque (i.e., $X_{DA} + X_{EA} + X_{FA}$) should now be permitted to be smaller than the total demand. The demand constraint for this warehouse should therefore be written as

$$X_{DA} + X_{EA} + X_{FA} \leq 350 \qquad \text{(Albuquerque demand)}$$

Likewise, the demand constraints at the Boston and Cleveland warehouses would need to be written as

$$X_{DB} + X_{EB} + X_{FB} \leq 200 \qquad \text{(Boston demand)}$$
$$X_{DC} + X_{EC} + X_{FC} \leq 200 \qquad \text{(Cleveland demand)}$$

In this case, because the supply constraints at the three factories will continue to be written as = constraints, the solution will show movements of 700 desks between the factories and the warehouses. The remaining demand of 50 desks will remain unsatisfied, and one or more of the warehouses will not get its full share of desks.

Use of a Dummy Location to Balance an Unbalanced Model

Some students may find it confusing to deal with ≤ and ≥ signs in the net flow constraints for unbalanced models. One way to avoid this is to use a dummy location to transform an unbalanced model into a balanced model. We illustrate this by revisiting the situation we just considered, where the total supply is 750 desks while the total demand is only 700 desks. We can covert this to a balanced model by creating a dummy demand node with a demand of 50 desks (i.e., the excess supply). Likewise, if the total demand is larger than the total supply, we can covert this to a balanced model by creating a dummy supply node with a supply value equal to the excess demand. All supply and demand constraints can then be written with = signs in the model.

What about the unit objective coefficients (e.g., shipping costs) from the dummy location? Since the dummy location doesn't actually exist, we need to make sure the solution focuses on flows from or to locations that actually exist and that it uses the dummy location only as a last resort to balance the flows. An easy way to accomplish this is to assign an objective coefficient of ∞ (for minimization models) or −∞ (for maximization models) to flows from or to the dummy location.[3]

Alternate Optimal Solutions

It is quite common for transportation models to have alternate optimal solutions.

Just as with regular LP problems, it is possible for a transportation model to have alternate or multiple optimal solutions. In fact, having multiple optimal solutions is quite common in transportation models. Practically speaking, multiple optimal solutions provide management with greater flexibility in selecting and using resources. Chapter 4 (section 4.4, on page 138) indicates that if the allowable increase or allowable decrease for the objective coefficient of a variable has a value of zero (in the Variable Cells table of the Solver Sensitivity Report), this usually indicates the presence of alternate optimal solutions. In Solved Problem 4-1 (see page 145), we saw how Solver can be used to identify alternate optimal solutions.

An Application of the Transportation Model: Facility Location

Deciding where to locate a new facility within an overall distribution system is aided by the transportation model.

The transportation model has proved to be especially useful in helping firms decide where to locate a new factory or warehouse. Because a new location has major financial implications for a firm, several alternative locations must usually be considered and evaluated. Even though a firm may consider a wide variety of subjective factors, including quality of labor supply, presence of labor unions, community attitude, utilities, and recreational and educational facilities,

[3] ∞ here simply means a value that is very large when compared to other objective coefficients in the model. For example, if the other objective coefficients have values such as \$4, \$5, etc., an objective coefficient of \$200 will be sufficient to represent ∞.

IN ACTION Answering Warehousing Questions at San Miguel Corporation

San Miguel Corporation, based in the Philippines, faces unique distribution challenges. With more than 300 products, including beer, alcoholic drinks, juices, bottled water, feeds, poultry, and meats to be distributed to every corner of the Philippine archipelago, shipping and warehousing costs make up a large part of total produce cost.

The company grappled with these questions:

- Which products should be produced in each plant, and in which warehouse should they be stored?
- Which warehouses should be maintained and where should new ones be located?
- When should warehouses be closed or opened?
- Which demand centers should each warehouse serve?

Turning to the transportation model, San Miguel was able to answer these questions. The firm used these types of warehouses: company owned and staffed, rented but company staffed, and contracted out (i.e., not company owned or staffed).

San Miguel's Operations Research Department computed that the firm saves $7.5 million annually with optimal beer warehouse configurations over the existing national configurations. In addition, analysis of warehousing for ice cream and other frozen products indicated that the optimal configuration of warehouses, compared with existing setup, produced a $2.17 million savings.

Source: Based on E. del Rosario. "Logistical Nightmare," *OR/MS Today* 26, 2 (April 1999): 44–46. Reprinted with permission.

a final decision also involves minimizing total production and shipping costs. This means that the **facility location analysis** should analyze each location alternative within the framework of the overall distribution system. The new location that will yield the minimum cost for the entire system should be the one recommended.

We solve a separate transportation model with each location to find the location with the lowest system cost.

How do we use the transportation model to help in this decision-making process? Consider a firm that is trying to decide between several competing locations for a new factory. To determine which new factory yields the lowest total systemwide cost, we solve separate transportation models: one for each of the possible locations. We illustrate this application of the transportation model in Solved Problem 5-1 at the end of this chapter, with the case of the Hardgrave Machine Company, which is trying to decide between Seattle, Washington, and Birmingham, Alabama, as a site to build a new factory.

5.4 Transportation Models with Max-Min and Min-Max Objectives

***Max-Min** and **Min-Max** models seek to reduce the variability in the values of the decision variables.*

In all LP models discussed so far (starting in Chapter 2), the objective function has sought to either maximize or minimize some function of the decision variables. However, there are some situations, especially in transportation settings, where we may be interested in a special type of objective function. The objective in these situations may be to *maximize the minimum value* of the decision variables (**Max-Min model**), or, alternatively, *minimize the maximum value* of the decision variables (**Min-Max model**). These types of objective functions are applicable when we want to reduce the variability in the values of the decision variables. Let us illustrate this issue by revisiting Executive Furniture's transportation example and formulating it as a Min-Max model.

Managers at Executive Furniture noticed that the optimal solution to their transportation problem (Screenshot 5-2B) recommends the use of only five of the nine available shipping routes. Further, the entire demand at Boston is being satisfied by the factory at Evansville. This means that if, for any reason, the Evansville factory has a production problem in a given month, the Boston warehouse is severely affected. To avoid this situation, the managers would like to distribute the shipments among all shipping routes, to the extent possible. They would like to achieve this by minimizing the maximum amount shipped on any specific route. Note that because the total number of desks available is fixed, if we reduce the number shipped on any route, shipments on some other route(s) will automatically increase. This implies that the difference between the largest and smallest shipments will be lowered to the extent allowed by the other constraints in the model.

To achieve this new objective, the managers are willing to allow an increase of up to 5% in the current total transportation cost. Because the current plan costs \$3,900 (Screenshot 5-2B), this implies that the maximum transportation cost allowed is $1.05 \times \$3{,}900 = \$4{,}095$.

FORMULATING THE PROBLEM To formulate this Min-Max model, we define a new decision variable as follows:

$$S = \text{maximum quantity shipped on any route}$$

The objective function is to minimize the value of S. We then set S to be greater than or equal to all the other decision variables (i.e., the nine shipping quantities) in the model. Note that doing so implies that S will be greater than the largest of the nine shipping amounts. Further, because S is being minimized in the objective function, S will automatically equal the maximum quantity shipped on any route. The complete LP model may be written as

$$\text{Minimize } S$$

subject to the constraints

$$\begin{array}{lll}
-X_{DA} - X_{DB} - X_{DC} & = -100 & \text{(Des Moines supply)} \\
-X_{EA} - X_{EB} - X_{EC} & = -300 & \text{(Evansville supply)} \\
-X_{FA} - X_{FB} - X_{FC} & = -300 & \text{(Fort Launderdale supply)} \\
X_{DA} + X_{EA} + X_{FA} & = 300 & \text{(Albuquerque demand)} \\
X_{DB} + X_{EB} + X_{FB} & = 200 & \text{(Boston demand)} \\
X_{DC} + X_{EC} + X_{FC} & = 200 & \text{(Cleveland demand)} \\
5X_{DA} + 4X_{DB} + 3X_{DC} & & \\
\quad + 8X_{EA} + 4X_{EB} + 3X_{EC} & & \\
\quad + 9X_{FA} + 7X_{FB} + 5X_{FC} & \leq 4{,}095 & \text{(Cost contraint)}
\end{array}$$

$$\begin{array}{lll}
S \geq X_{DA} & S \geq X_{DB} & S \geq X_{DC} \\
S \geq X_{EA} & S \geq X_{EB} & S \geq X_{EC} \\
S \geq X_{FA} & S \geq X_{FB} & S \geq X_{FC}
\end{array}$$

$$\text{All variables} \geq 0$$

File: 5-3.xls

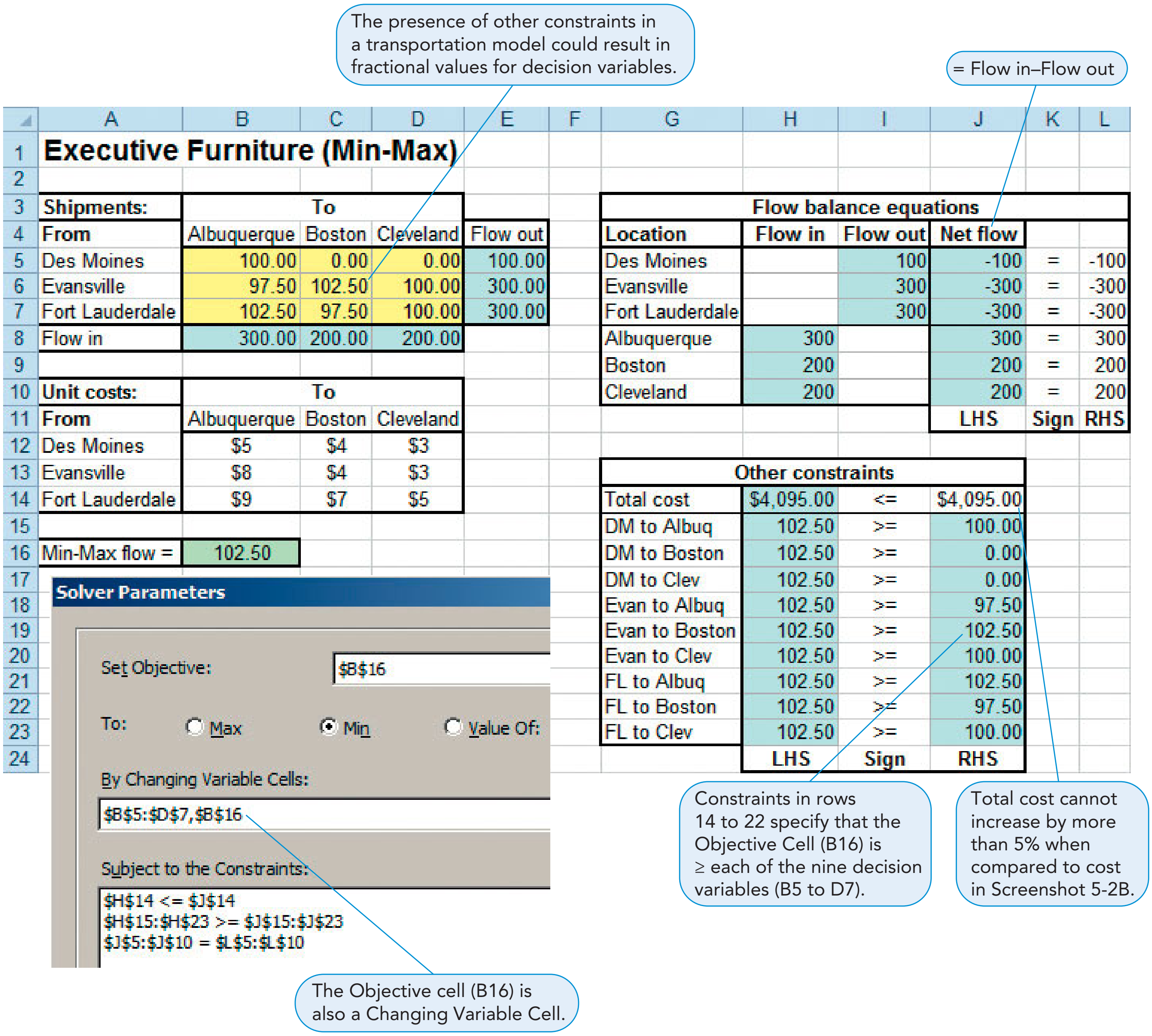
SOLVING THE PROBLEM The Excel layout and Solver entries for Executive Furniture's revised transportation model are shown in Screenshot 5-3. Notice that the objective cell (B16) is also a decision variable (i.e., changing variable cell). The flow balance equations are the same as in Screenshot 5-2B. However, we now have a new constraint on the total cost, as well as constraints specifying that cell B16 should be greater than or equal to each of the nine shipping routes (cells B5:D7).

The addition of constraints other than supply and demand constraints may cause the optimal solution in network flow models to no longer be integers.

INTERPRETING THE RESULTS The revised optimal solution costs \$4,095 and uses seven of the nine available shipping routes. The maximum number shipped on any route drops from 200 (in Screenshot 5-2B) to only 102.50. This solution reveals an interesting point. Recall that if all supply and demand values are integers, all flows are also integer values in network flow models. However, when we add additional constraints (i.e., other than supply and demand constraints), this integer solution property is destroyed, and the resulting solution can have fractional values. In Executive Furniture's case, the managers would need to round up and round down values appropriately to achieve an integer-valued shipping plan.

The LP model can be set up in a similar manner if the objective is to maximize the minimum value of the decision variables (i.e., Max-Min model). The only modifications needed to the Min-Max model discussed previously are to (1) change the objective from Max to Min and (2) change the sign in the constraints linking the variable S to the decision variables from $\geq$ to $\leq$.

SCREENSHOT 5-3 **Excel Layout and Solver Entries for Executive Furniture—Min-Max**



Executive Furniture (Min-Max)

Shipments:	To			
From	Albuquerque	Boston	Cleveland	Flow out
Des Moines	100.00	0.00	0.00	100.00
Evansville	97.50	102.50	100.00	300.00
Fort Lauderdale	102.50	97.50	100.00	300.00
Flow in	300.00	200.00	200.00	

Unit costs:	To		
From	Albuquerque	Boston	Cleveland
Des Moines	$5	$4	$3
Evansville	$8	$4	$3
Fort Lauderdale	$9	$7	$5

Min-Max flow =	102.50

Flow balance equations					
Location	Flow in	Flow out	Net flow		
Des Moines		100	-100	=	-100
Evansville		300	-300	=	-300
Fort Lauderdale		300	-300	=	-300
Albuquerque	300		300	=	300
Boston	200		200	=	200
Cleveland	200		200	=	200
			LHS	Sign	RHS

Other constraints			
Total cost	$4,095.00	<=	$4,095.00
DM to Albuq	102.50	>=	100.00
DM to Boston	102.50	>=	0.00
DM to Clev	102.50	>=	0.00
Evan to Albuq	102.50	>=	97.50
Evan to Boston	102.50	>=	102.50
Evan to Clev	102.50	>=	100.00
FL to Albuq	102.50	>=	102.50
FL to Boston	102.50	>=	97.50
FL to Clev	102.50	>=	100.00
	LHS	Sign	RHS

5.5 Transshipment Model

Transshipment models include nodes that can have shipments arrive as well as leave.

In the basic transportation model, shipments either flow *out* of supply nodes or flow *in* to demand nodes. That is, it is possible to explicitly distinguish between supply nodes and demand nodes for flows. In the more general form of the transportation model, called the transshipment model, flows can occur both *out* of and *in* to the same node in three ways:

1. If the total flow *in* to a node is less than the total flow *out* of the node, the node then represents a net creator of goods—that is, a supply node. The flow balance equation for this type of node will therefore have a negative RHS value.
2. If the total flow *in* to a node exceeds the total flow *out* of the node, the node then represents a net consumer of goods—that is, a demand node. The flow balance equation for this type of node will therefore have a positive RHS value.
3. If the total flow *in* to a node is equal to the total flow *out* of the node, the node then represents a pure transshipment node. The flow balance equation for this type of node will therefore have a zero RHS value.

We use two examples in the following sections to illustrate transshipment models. The first is a simple extension of the Executive Furniture Company's transportation model. The second is a larger example that includes some pure transshipment nodes.

Executive Furniture Company Example—Revisited

Transshipment example: A modified form of the Executive Furniture example.

Let us consider a modified version of the Executive Furniture Company example from section 5.3. As before, the company has factories in Des Moines, Evansville, and Fort Lauderdale and warehouses in Albuquerque, Cleveland, and Boston. Recall that the supply at each factory and demand at each warehouse are shown in Figure 5.3 on page 166.

Now suppose that due to a special contract with an Evansville-based shipping company, it is possible for Executive Furniture to ship desks from its Evansville factory to its three warehouses at very low unit shipping costs. These unit costs are so attractive that Executive Furniture is considering shipping all the desks produced at its other two factories (Des Moines and Fort Lauderdale) to Evansville and then using this new shipping company to move desks from Evansville to all its warehouses.

The revised unit shipping costs are shown in Table 5.2. Note that the Evansville factory now shows up both in the "From" and "To" entries because it is possible for this factory to receive desks from other factories and then ship them out to the warehouses. There are therefore two additional shipping routes available: Des Moines to Evansville and Fort Lauderdale to Evansville.

LP Formulation for Executive Furniture's Transshipment Model

The LP formulation for this model follows the same logic and structure as the formulation for Executive Furniture's transportation model (see section 5.3). However, we now have two *additional* decision variables for the two new shipping routes. We define these as follows:

$$X_{DE} = \text{number of desks shipped from Des Moines to Evansville}$$
$$X_{FE} = \text{number of desks shipped from Fort Lauderdale to Evansville}$$

OBJECTIVE FUNCTION The objective function for this transshipment model, including the two additional decision variables and using the unit costs shown in Table 5.2, can be written as follows:

$$\begin{aligned}\text{Minimize total shipping costs} = {} & 5X_{DA} + 4X_{DB} + 3X_{DC} + 2X_{DE} + 3X_{EA} + 2X_{EB} \\ & + X_{EC} + 9X_{FA} + 7X_{FB} + 5X_{FC} + 3X_{FE}\end{aligned}$$

CONSTRAINTS Once again, we need to write flow balance constraints for each node in the network. Let us first consider the net flows at the Des Moines and Fort Lauderdale factories. After taking into account the desks shipped from either of these locations to the Evansville factory (rather than directly to the warehouses), the relevant flow balance equations can be written as

$$(0) - (X_{DA} + X_{DB} + X_{DC} + X_{DE}) = -100 \qquad \text{(Des Moines supply)}$$
$$(0) - (X_{FA} + X_{FB} + X_{FC} + X_{FE}) = -300 \qquad \text{(Fort Lauderdale supply)}$$

As usual, supplies have been expressed as negative numbers on the RHS. Now, let us model the flow equation at Evansville:

$$\begin{aligned}\text{New flow at Evansville} &= (\text{Total flow } \textit{in} \text{ to Evansville}) - (\text{Total flow } \textit{out} \text{ of Evansville}) \\ &= (X_{DE} + X_{FE}) - (X_{EA} + X_{EB} + X_{EC})\end{aligned}$$

TABLE 5.2
Revised Transportation Costs per Desk for Executive Furniture

	TO			
	ALBUQUERQUE	BOSTON	CLEVELAND	EVANSVILLE
Des Moines	\$5	\$4	\$3	\$2
Evansville	\$3	\$2	\$1	—
Fort Lauderdale	\$9	\$7	\$5	\$3

This net flow is equal to the total number of desks produced—namely, the supply, at Evansville (which would also appear as a negative number in the flow balance constraint). Therefore,

$$\text{New flow at Evansville} = (X_{DE} + X_{FE}) - (X_{EA} + X_{EB} + X_{EC}) = -300$$

There is no change in the demand constraints that represent the warehouse requirements. So, as discussed in section 5.3, they are

$$X_{DA} + X_{EA} + X_{FA} = 300 \qquad \text{(Albuquerque demand)}$$
$$X_{DB} + X_{EB} + X_{FB} = 200 \qquad \text{(Boston demand)}$$
$$X_{DC} + X_{EC} + X_{FC} = 200 \qquad \text{(Cleveland demand)}$$

File: 5-4.xls

EXCEL SOLUTION Screenshot 5-4, which uses the tabular layout for representing the network flows, shows the Excel layout and Solver entries for Executive Furniture's transshipment model. Note that the net flow at Evansville (cell K6) is calculated as (cell I6 – cell J6), where cell I6 (= cell E8) is the total flow *in* to the Evansville factory, and cell J6 (= cell F6) represents the total flow *out* of Evansville. The difference of 300 is the supply of desks created at the Evansville factory.

In the revised solution, which now has a total transportation cost of $2,600, Executive should ship the 300 desks made at Fort Lauderdale to Evansville and then ship the consolidated load to the warehouses. It continues, though, to be cost beneficial to ship desks made at the Des Moines factory directly to a warehouse.

Lopez Custom Outfits—A Larger Transshipment Example

Paula Lopez makes and sells custom outfits for theme parties hosted by her wealthy clients. She uses three tailoring shops located in the Northeastern United States (in Albany, Boston, and Hartford) to make the outfits. These are then shipped to her finishing facilities in Charlotte and Richmond, where they are further customized to suit the clients' specifications and inspected before being shipped to the clients. Paula's clients are based primarily in four cities: Dallas, Louisville, Memphis, and Nashville.

SCREENSHOT 5-4 Excel Layout and Solver Entries for Executive Furniture—Transshipment

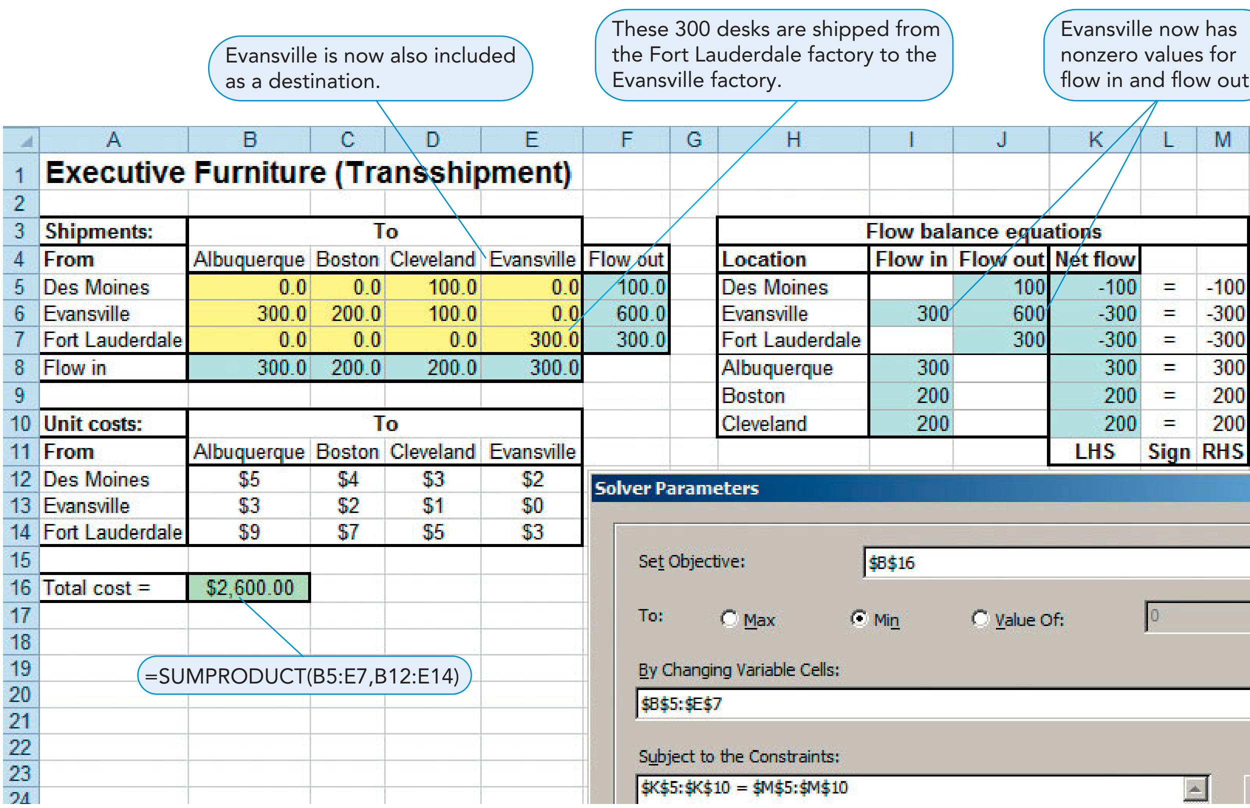


TABLE 5.3 TRANSPORTATION COSTS PER OUTFIT FOR LOPEZ CUSTOM OUTFITS

	TO	
FROM	CHARLOTTE	RICHMOND
Albany	\$40	\$55
Boston	\$43	\$46
Hartford	\$50	\$50

	TO			
FROM	DALLAS	LOUISVILLE	MEMPHIS	NASHVILLE
Charlotte	\$38	\$40	\$51	\$40
Richmond	\$30	\$47	\$41	\$45

Paula has received the following firm orders for custom outfits: 450 from Dallas, 300 from Louisville, 275 from Memphis, and 400 from Nashville. Her tailoring shop in Albany can make up to 450 outfits, the Boston shop can handle up to 500 outfits, and the Hartford shop has the capacity to make 580 outfits. Therefore, Paula's total tailoring capacity exceeds the total demand for outfits. She therefore knows that she will be able to fully satisfy all her clients' needs.

There is no production cost difference between the three tailoring shops, and Paula sells all her outfits at the same fixed price to her clients. There is also no cost difference between Charlotte and Richmond with regard to the customization and inspection processes at these locations. Further, since these processes do not consume too much time or space, Paula need not be concerned about both these locations with regard to capacity. Paula's cost difference between the various locations arises primarily from the shipping costs per outfit, which are summarized in Table 5.3.

LP Formulation for Lopez Custom Outfits Transshipment Model

Before we discuss the LP formulation for this model, it is useful to draw it as a network so that we can visualize the various flows that could occur. Figure 5.4 shows the network for Paula's transshipment model. We note that there are arcs both coming into and going out of the transshipment nodes, Charlotte and Richmond.

The LP formulation for this model involves 14 decision variables, 1 for each of the arcs shown in Figure 5.4. Let us define these decision variables as

$$X_{ij} = \text{number of outfits shipped from location } i \text{ to location } j$$

where

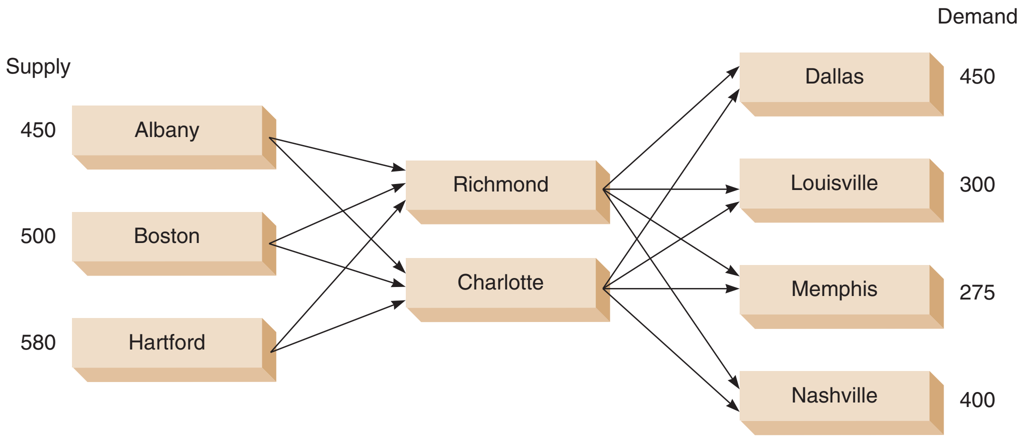
i = A (for Albany), B (for Boston), H (for Hartford), C (for Charlotte), or R (for Richmond)

j = C (for Charlotte), R (for Richmond), D (for Dallas), L (for Louisville), M (for Memphis), or N (for Nashville)

OBJECTIVE FUNCTION The objective function for this problem seeks to minimize the total transshipment cost and can be expressed as

$$\text{Minimize } \$40X_{AC} + \$55X_{AR} + \$43X_{BC} + \$46X_{BR} + \$50X_{HC} + \$50X_{HR} + \\ \$38X_{CD} + \$40X_{CL} + \$51X_{CM} + \$40X_{CN} + \$30X_{RD} + \$47X_{RL} + \$41X_{RM} + \$45X_{RN}$$

FIGURE 5.4
Network Model for Lopez Custom Outfits—Transshipment



CONSTRAINTS Here again, we need to write flow balance constraints for each of the nine nodes in the network (see Figure 5.4). For the three tailoring shops, the net flow equations can be expressed as

$$
\begin{aligned}
(0) - (X_{AC} + X_{AR}) &\geq -450 && \text{(Albany supply)}\\
(0) - (X_{BC} + X_{BR}) &\geq -500 && \text{(Boston supply)}\\
(0) - (X_{HC} + X_{HR}) &\geq -580 && \text{(Hartford supply)}
\end{aligned}
$$

As usual, supplies have been expressed as negative numbers on the RHS. Also, because total supply exceeds total demand, this is an unbalanced model in which not all tailoring shops will be used to their full capacity. The supply constraints therefore need to be written as inequalities. Recall from our discussion of unbalanced models on page 170 that based on the definition of net flows being used in this textbook, we should express the supply constraints in such situations using the $\geq$ sign.

Now, let us model the flow equation at the two transshipment locations, Charlotte and Richmond:

$$
\begin{aligned}
(X_{AC} + X_{BC} + X_{HC}) - (X_{CD} + X_{CL} + X_{CM} + X_{CN}) &= 0 && \text{(net flow at Charlotte)}\\
(X_{AR} + X_{BR} + X_{HR}) - (X_{RD} + X_{RL} + X_{RM} + X_{RN}) &= 0 && \text{(net flow at Richmond)}
\end{aligned}
$$

Note that the RHS of both these equations is zero since Charlotte and Richmond are pure transshipment locations and all outfits simply pass through these locations (i.e., no outfit is created there, and none are consumed there).

Finally, we model the net flow equations at the four demand locations as

$$
\begin{aligned}
(X_{CD} + X_{RD}) - (0) &= 450 && \text{(Dallas demand)}\\
(X_{CL} + X_{RL}) - (0) &= 300 && \text{(Louisville demand)}\\
(X_{CM} + X_{RM}) - (0) &= 275 && \text{(Memphis demand)}\\
(X_{CN} + X_{RN}) - (0) &= 400 && \text{(Nashville demand)}
\end{aligned}
$$

Since all demands will be met in this model, we can express all four demand constraints using = signs.

File: 5-5.xls

EXCEL SOLUTION Screenshot 5-5 shows the Excel layout and Solver entries for Lopez's transshipment model. Note that Charlotte and Richmond appear both in the rows and in the columns. Also, even though the table of flows contains 30 cells (i.e., cells B5:G9), only 14 of these are actual routes that exist (shown in yellow in Screenshot 5-5).

How do we enter these changing variable cells in Solver? There are two simple ways of doing this. First, we can specify only the 14 shaded cells as changing variable cells. Recall that to do this in Solver, we must separate entries for nonadjacent cells by using commas (i.e., B5:C7, D8:G9). Although this approach is quite straightforward, it could be cumbersome, especially if there are many nonadjacent decision variables in the model.

The second (and easier) approach is to specify the entire cell range B5:G9 in the By Changing Variable Cells box in Solver. Then, for all routes that do not exist (e.g., Albany to Dallas), we simply set the unit cost to artificially high values ($2,000 in this case). We have illustrated this approach in Screenshot 5-5.

The model includes three locations with negative RHS values (tailors), two locations with zero RHS values (Charlotte and Richmond), and four locations with positive RHS values (demands).

The optimal solution has a total transshipment cost of $116,775. Paula should use the Albany and Boston tailoring shops to full capacity but use the Hartford tailoring shop for only 475 units of their 580 capacity. Shipments from Albany and Hartford go to just one of the transshipment locations, but shipments from Boston are split between the two locations. Each demand location then receives all its outfits from one of the two transshipment locations.

SCREENSHOT 5-5 Excel Layout and Solver Entries for Lopez Custom Outfits—Transshipment

Only the 16 shaded cells represent actual shipping routes.

Hartford is not being used to full capacity.

Lopez Custom Outfits (Transshipment)

Shipments:	To						
From	Charlotte	Richmond	Dallas	Louisville	Memphis	Nashville	Flow out
Albany	450.0	0.0	0.0	0.0	0.0	0.0	450.0
Boston	250.0	250.0	0.0	0.0	0.0	0.0	500.0
Hartford	0.0	475.0	0.0	0.0	0.0	0.0	475.0
Charlotte	0.0	0.0	0.0	300.0	0.0	400.0	700.0
Richmond	0.0	0.0	450.0	0.0	275.0	0.0	725.0
Flow in	700.0	725.0	450.0	300.0	275.0	400.0	

Unit costs:	To					
From	Atlanta	Richmond	Dallas	Louisville	Memphis	Nashville
Albany	$40	$55	$2,000	$2,000	$2,000	$2,000
Boston	$43	$46	$2,000	$2,000	$2,000	$2,000
Hartford	$50	$50	$2,000	$2,000	$2,000	$2,000
Charlotte	$0	$2,000	$38	$40	$51	$40
Richmond	$2,000	$0	$30	$47	$41	$45

Total cost = $116,775

Flow balance equations					
Location	Flow in	Flow out	Net flow		
Albany		450	-450	>=	-450
Boston		500	-500	>=	-500
Hartford		475	-475	>=	-580
Charlotte	700	700	0	=	0
Richmond	725	725	0	=	0
Dallas	450		450	=	450
Louisville	300		300	=	300
Memphis	275		275	=	275
Nashville	400		400	=	400
			LHS	Sign	RHS

=SUMPRODUCT(B5:G9, B14:G18)

A high unit cost (like $2,000 in this case) prevents shipments on these routes.

Solver Parameters

Set Objective: $B$20

To: Max Min Value Of: 0

By Changing Variable Cells:
$B$5:$G$9

Subject to the Constraints:
$M$5:$M$7 >= $O$5:$O$7
$M$8:$M$13 = $O$8:$O$13

The entire table B5:C9 can be entered as the Changing Variable Cells even though some of the routes (non-shaded) do not exist.

Charlotte and Richmond are pure transshipment nodes with RHS=0.

IN ACTION British Telecommunications Uses the Assignment Model to Schedule Field Engineers

British Telecommunications PLC (BT) employs over 50,000 field engineers across the UK to repair network faults, maintain networks, and provide other services to customers. It is critical for BT to optimize its field workforce scheduling, which involves sending the right engineer to the right customer at the right time and place with the right equipment. BT's ability to provide high quality service while achieving maximum productivity and low operational costs is vital to its success.

BT's automated work-management and field-communication system, called Work Manager, is equipped with a real-time allocation algorithm that uses an assignment model to determine the minimal-cost assignment set, The model considers the skill requirements of individual engineers as well as the geographic distribution of the available engineers and job sites. BT runs an extended version of Work Manager every 5 to 15 minutes, with a typical problem consisting of 50 to 100 engineers and 300 to 500 tasks. The model usually returns a good schedule within one minute.

BT estimates that the use of Work Manager has saved them $150 million annually even with only 28,000 engineers scheduled using the model. The amount would exceed $250 million annually when it fully covers all engineers. BT has also improved its performance with regards to due dates, appointment windows, and skill requirements.

Source: Based on D. Lesaint, C. Voudouris, and N. Azarmi. "Dynamic Workforce Scheduling for British Telecommunications PLC," *Interfaces* 30, 1 (January–February 2000): 45–56.

5.6 Assignment Model

The next model we study is the assignment model. Recall from section 5.1 that this model seeks to identify an optimal one-to-one assignment of people to tasks, jobs to machines, and so on. The typical objective is to minimize the total cost of the assignment, although a maximization objective is also possible. To represent each assignment model, we associate a table. Generally, the rows denote the people or jobs we want to assign, and the columns denote the tasks or machines to which we want them assigned. The numbers in the table are the costs (or benefits) associated with each particular one-to-one assignment.

Fix-It Shop Example

Assignment example: Fix-It Shop

As an illustration of the assignment model, let us consider the case of Fix-It Shop, which has just received three new rush projects to repair: (1) a radio, (2) a toaster oven, and (3) a coffee table. Three workers, each with different talents and abilities, are available to do the jobs. The Fix-It Shop owner estimates what it will cost in wages to assign each of the workers to each of the three projects. The costs, which are shown in Table 5.4, differ because the owner believes that each worker will differ in speed and skill on these quite varied jobs.

The objective is to assign projects to people (one project to one person) so that the total costs are minimized.

The owner's objective is to assign the three projects to the workers in a way that will result in the lowest total cost to the shop. Note that the assignment of people to projects must be on a one-to-one basis; each project must be assigned to at most one worker only, and vice versa. If the number of rows in an assignment model is equal to the number of columns (as in the Fix-It example), we refer to this problem as a *balanced* assignment model.

One way to solve (small) assignment models is to enumerate all possible solutions.

Because the Fix-It Shop example consists of only three workers and three projects, one easy way to find the best solution is to list all possible assignments and their respective costs. For example, if Adams is assigned to project 1, Brown to project 2, and Cooper to project 3, the total cost will be $11 + $10 + $7 = $28. Table 5.5 summarizes all six assignment options. The table also shows that the least-cost solution would be to assign Cooper to project 1, Brown to project 2, and Adams to project 3, at a total cost of $25.

Obtaining solutions by enumeration works well for small models but quickly becomes inefficient as assignment models become larger. For example, a model involving the assignment of eight workers and eight tasks, which actually is not that large in a real-world situation, yields $8!(= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)$, or 40,320 possible solutions! Because it would clearly be impractical to individually examine so many alternatives, a more efficient solution approach is needed.

TABLE 5.4
Estimated Project Repair Costs for Fix-It Shop

	PROJECT		
PERSON	1	2	3
Adams	$11	$14	$ 6
Brown	$ 8	$10	$11
Cooper	$ 9	$12	$ 7

TABLE 5.5
Summary of Fix-It Shop's Assignment Alternatives and Costs

PROJECT ASSIGNMENT					
1	2	3	LABOR COSTS ($)		TOTAL COSTS
Adams	Brown	Cooper	$11 + $10 + $ 7	=	$28
Adams	Cooper	Brown	$11 + $12 + $11	=	$34
Brown	Adams	Cooper	$ 8 + $14 + $ 7	=	$29
Brown	Cooper	Adams	$ 8 + $12 + $ 6	=	$26
Cooper	Adams	Brown	$ 9 + $14 + $11	=	$34
Cooper	Brown	Adams	$ 9 + $10 + $ 6	=	$25

Solving Assignment Models

Each supply and each demand in an assignment model equals one unit.

A straightforward approach to solving assignment models is to formulate them as a transportation model. To do so for the Fix-It Shop problem, let us view each worker as a supply node in a transportation network with a supply of one unit. Likewise, let us view each project as a demand node in the network with a demand of one unit. The arcs connecting the supply nodes to the demand nodes represent the possible assignment of a supply (worker) to a demand (project). The network model is illustrated in Figure 5.5.

We see that this network looks identical to a transportation model with three supply nodes and three demand nodes. But here, each supply and each demand is equal to one unit. The objective is to find the least-cost solution that uses the one-unit supplies at the origin nodes to satisfy the one-unit demands at the demand nodes. However, we need to also ensure that each worker *uniquely* gets assigned to just one project, and vice versa. That is, the *entire* supply of one unit at an origin node (worker) should flow to the same demand node (project), indicating the assignment of a worker to a project. How do we ensure this? The answer lies in the special property of network models stated earlier: When all the supplies and demands in a network model are whole numbers (as in this case), the resulting solution will automatically have integer-valued flows on the arcs.

The special integer flow property of network models automatically ensures unique assignments.

Consider the "flow" out of the supply node for Adams in the Fix-It Shop example. The three arcs (to projects 1, 2, and 3) denote the assignment of Adams to these projects. Due to the integer property of the resulting network flows, the only possible solutions will have a flow of 1 on one of the three arcs and a flow of 0 on the other two arcs. This is the only way in which a total flow of 1 (equal to the "supply" at the node representing Adams) can flow on these arcs and have integer values. The arc that has a flow of 1 in the optimal solution will indicate the project to which Adams should be assigned. Likewise, arcs that have flows of 1 and originate from the other two supply nodes will show the optimal assignments for those two workers.

Even without us constraining it to be so, the solution to the assignment model yields a solution in which the optimal values of the decision variables are either 1 (indicating the assignment of a worker to a project) or 0 (indicating that the worker should not be assigned to the project). In fact, there are several situations in which such decision variables, known as *binary*, or 0–1, variables must have values of zero or one in the formulation itself. We study these types of models in more detail in Chapter 6.

LP Formulation for Fix-It Shop's Assignment Model

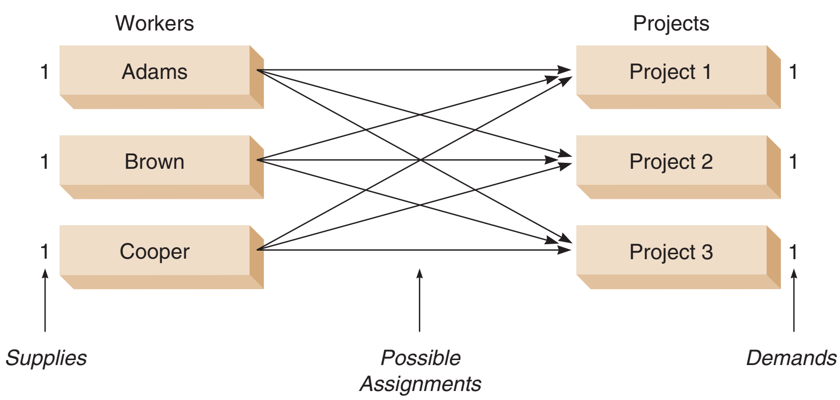
We now develop the LP model for Fix-It Shop's example. Let

X_{ij} = "Flow" on arc from node denoting worker i to node denoting project j. The solution value will equal 1 if worker i is assigned to project j, and will equal 0 otherwise.

where

$$i = \text{A (for Adams), B (for Brown), C (for Cooper)}$$
$$j = 1 \text{ (for project 1), 2 (for project 2), 3 (for project 3)}$$

FIGURE 5.5
Network Model for Fix-It Shop—Assignment



OBJECTIVE FUNCTION The objective is to minimize the total cost of assignment and is expressed as

$$\text{Minimize total assignment costs} = \$11X_{A1} + \$14X_{A2} + \$6X_{A3} + \$8X_{B1} + \$10X_{B2} + \$11X_{B3} + \$9X_{C1} + \$12X_{C2} + \$7X_{C3}$$

Here again, we write supply constraints and demand constraints.

CONSTRAINTS As in the transportation model, we have supply constraints at each of the three supply nodes (workers) and demand constraints at each of the three demand nodes (projects). Using the standard convention we have adopted for all flow balance equations, these can be written as

$$\begin{aligned}
-X_{A1} - X_{A2} - X_{A3} &= -1 && \text{(Adams availability)}\\
-X_{B1} - X_{B2} - X_{B3} &= -1 && \text{(Brown availability)}\\
-X_{C1} - X_{C2} - X_{C3} &= -1 && \text{(Cooper availability)}\\
X_{A1} + X_{B1} + X_{C1} &= 1 && \text{(project 1 requirement)}\\
X_{A2} + X_{B2} + X_{C3} &= 1 && \text{(project 2 requirement)}\\
X_{A3} + X_{B3} + X_{C3} &= 1 && \text{(project 3 requirement)}
\end{aligned}$$

File: 5-6.xls

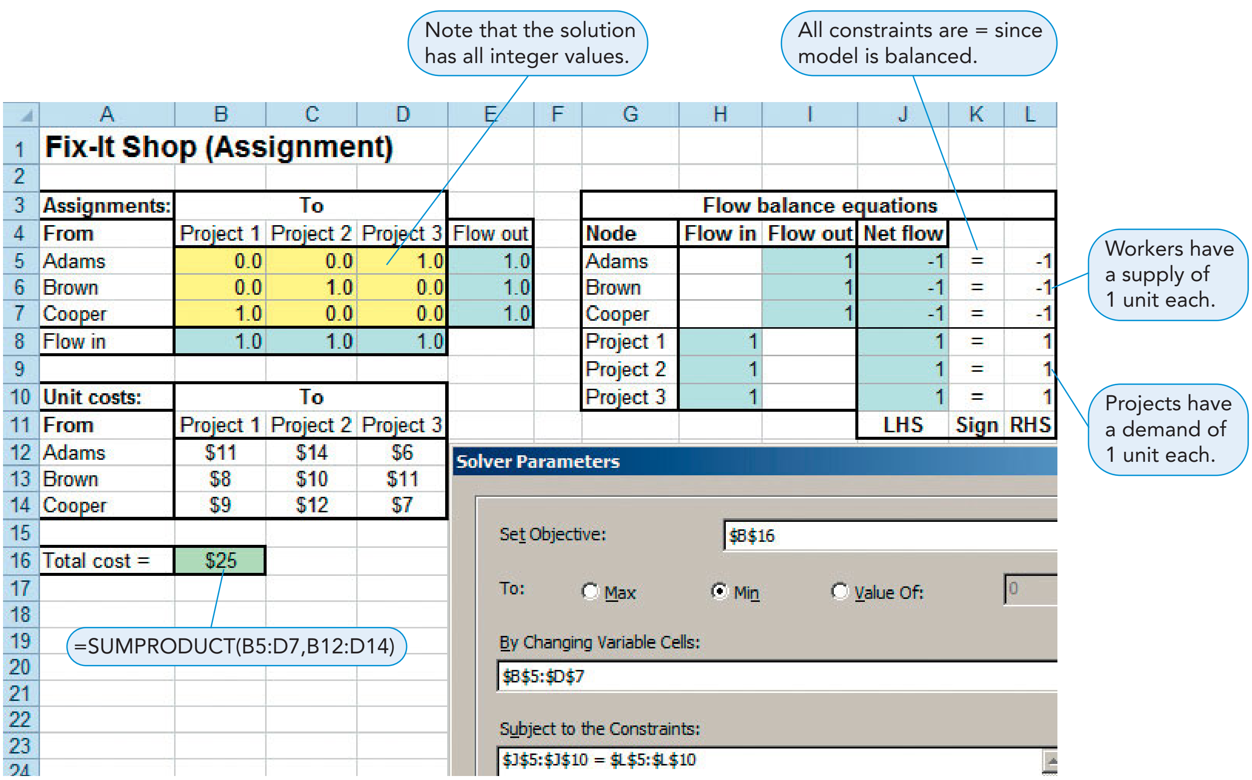
EXCEL SOLUTION Screenshot 5-6 shows the Excel layout and Solver entries for Fix-It Shop's assignment model. The optimal solution identified by the model indicates that Adams should be assigned to project 3, Brown to project 2, and Cooper to project 1, for a total cost of $25.

An assignment model can sometimes involve a maximization objective.

SOLVING MAXIMIZATION ASSIGNMENT MODELS The model discussed here can be very easily modified to solve *maximization* assignment models, in which the objective coefficients represent profits or benefits rather than costs. The only change needed would be in the statement of the objective function (which would be set to maximize instead of minimize).

UNBALANCED ASSIGNMENT MODELS In the Fix-It Shop example, the total number of workers equaled the total number of projects. All supply and demand constraints could therefore

SCREENSHOT 5-6 Excel Layout and Solver Entries for Fix-It Shop—Assignment



IN ACTION Scheduling the Belgian Soccer League with the Assignment Model

Worldwide interest in soccer has increased significantly over the past several years and Belgian soccer is no exception. Belgacom TV pays millions of Euros each year for the soccer broadcasting rights and there is a great deal of interest in ensuring that the league's schedule is properly designed. In addition to the obvious influence on the results of the sports competition, the schedule also affects game attendance, public interest, and the league's profitability and attractiveness to broadcasters, sponsors, and advertisers in subsequent years.

Until the 2005–2006 season schedules were created manually, which resulted in several teams viewing the schedules as being unbalanced and unfair. There were even accusations that the chairman of the calendar committee was favoring his own team. For the 2006–2007 season, the authors used an assignment model to develop the schedule for the Jupiler league, the highest division in Belgian Soccer. The league is organized as a double round-robin tournament with 18 teams with several constraints such as no team can play more than two consecutive home or away matches, the total number of breaks is minimal, and no team should start or end the season with a break. The authors expanded their model to a two-phased approach starting with the 2007–2008 season. In the first phase, each team is assigned a home-away pattern; in the second phase, the actual opponents are determined.

A spokesperson for the Belgian soccer league states that due to the use of this two-phased assignment model, "we have been able to come up with schedules that are much more satisfying for our partners (the police, Belgacom TV, and the clubs). In addition, the transparency of the process of agreeing upon a schedule has improved considerably as well."

Source: Based on D. Goossens and F. Spieksma. "Scheduling the Belgian Soccer League," *Interfaces* 39, 2 (March–April 2009): 109–118.

be specified as equalities (i.e., using the = sign). What if the number of workers exceeds the number of projects, or vice versa? In these cases, we have *unbalanced* assignment models and, just as in the case of unbalanced transportation models, the supply or demand constraints need to be modified accordingly. For example, if the number of workers exceeds the number of projects, the supply constraints would become inequalities, and the demand constraints would remain equality constraints. In contrast, if the number of projects exceeds the number of workers, the supply constraints would remain equality constraints, and the demand constraints would become inequalities. Solved Problem 5-2 at the end of this chapter shows an example of an unbalanced assignment model.

5.7 Maximal-Flow Model

A maximal-flow model finds the most that can flow through a network.

The *maximal-flow* model allows us to determine the maximum amount that can flow from a given origin node to a given destination node in a network with capacitated arcs. It has been used, for example, to find the maximum number of automobiles that can flow through a state highway road system.

Road System in Waukesha, Wisconsin

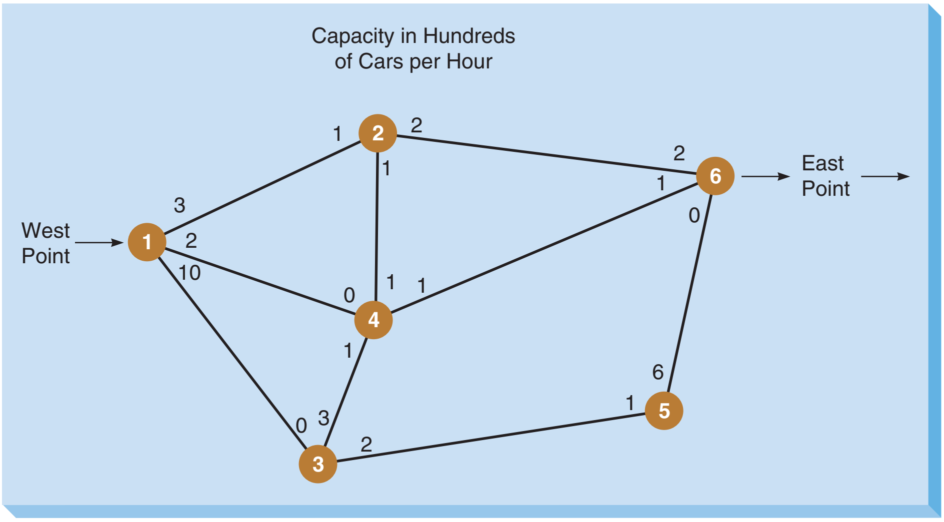
Waukesha, a small town in Wisconsin, is in the process of developing a road system for the downtown area. Bill Blackstone, a city planner, would like to determine the maximum number of cars that can flow through the town from west to east. The road network is shown in Figure 5.6, where the arcs represent the roads.

Traffic can flow in both directions.

The numbers by the nodes indicate the maximum number of cars (in hundreds of cars per hour) that can flow (or travel) *from* the various nodes. For example, the number 3 by node 1 (on the road from node 1 to node 2) indicates that 300 cars per hour can travel from node 1 to node 2. Likewise, the numbers 1, 1, and 2 by node 2 indicate that 100, 100, and 200 cars can travel per hour on the roads from node 2 to nodes 1, 4, and 6, respectively. Note that traffic can flow in both directions down a road. A zero (0) means no flow in that direction, or a one-way road.

Unlike the transportation and assignment models, in which there are multiple origin nodes and multiple destination nodes, the typical maximal-flow model has a single starting node (origin) and a single ending node (destination).

FIGURE 5.6
Road Network for Waukesha—Maximal-Flow



LP Formulation for Waukesha Road System's Maximal-Flow Model

We replace each two-way road (arc) with a pair of one-way roads.

To formulate this example as an LP model, we first replace each two-way (bidirectional) road in the network with two one-way (unidirectional) roads with flows in opposite directions. Note that some of the unidirectional roads (e.g., the road from node 4 to node 1, the road from node 6 to node 5) are not needed because the maximum flow permissible in that direction is zero (i.e., it is a one-way road). The revised network for Waukesha therefore has 15 unidirectional roads (i.e., roads $1 \rightarrow 2$, $1 \rightarrow 3$, $1 \rightarrow 4$, $2 \rightarrow 1$, $2 \rightarrow 4$, $2 \rightarrow 6$, $3 \rightarrow 4$, $3 \rightarrow 5$, $4 \rightarrow 2$, $4 \rightarrow 3$, $4 \rightarrow 6$, $5 \rightarrow 3$, $5 \rightarrow 6$, $6 \rightarrow 2$, and $6 \rightarrow 4$).

There is a decision variable associated with each arc in the network.

As with the transportation and assignment models, the presence of 15 unidirectional arcs in the network implies that there are 15 decision variables in Waukesha's maximal-flow model—1 for each arc (road) in the network. Let

$$X_{ij} = \text{Number of cars that flow (or travel) per hour on road from node } i \text{ to node } j$$

where i and j each equal 1, 2, 3, 4, 5, or 6. Of course, flow variables are defined only on roads that actually exist. For example, X_{12} (i.e., $i = 1, j = 2$) is defined, while X_{15} (i.e., $i = 1, j = 5$) is not defined.

We need to determine the maximum number of cars that can originate at node 1 and terminate at node 6. Hence, node 1 is the origin node in this model, and node 6 is the destination node. All other nodes (nodes 2 to 5) are transshipment nodes, where flows of cars neither start nor end. However, unlike in the transportation and assignment models, there is neither a known quantity of "supply" of cars available at node 1, nor is there a known quantity of "demand" for cars required at node 6. For this reason, we need to slightly modify the network to set up and solve the maximal-flow model using LP.

We add a one-way dummy road (arc) from the destination node to the source node.

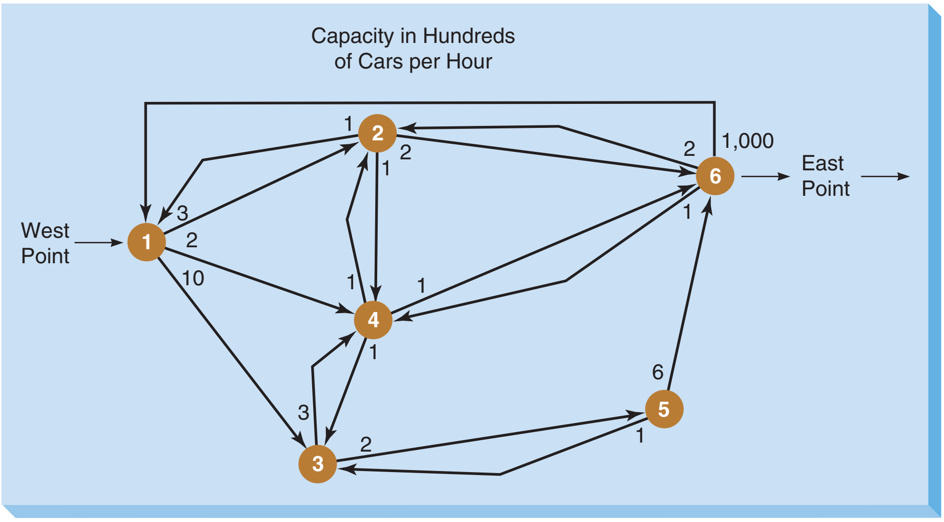
The modification consists of creating a unidirectional *dummy* arc (road) going *from* the destination node (node 6) *to* the origin node (node 1). We call this a dummy arc because the arc (road) really does not exist in the network and has been created only for modeling purposes. The capacity of this dummy arc is set at infinity (or any artificially high number, such as 1,000 for the Waukesha example). The modified network is shown in Figure 5.7.

OBJECTIVE FUNCTION Let us consider the objective function first. The objective is to maximize the total number of cars flowing *in* to node 6. Assume that there are an unknown number of cars flowing on the dummy road from node 6 to node 1. However, because there is no supply at node 6 (i.e., no cars are created at node 6), the entire number of cars flowing *out* of node 6 (on road $6 \rightarrow 1$) must consist of cars that flowed *in* to node 6. Likewise, because there is no demand at node 1 (i.e., no cars are consumed at node 1), the entire number of cars on road $6 \rightarrow 1$ must consist of cars that originally flowed *out* of node 1 (to nodes 2, 3, and 4).

The objective is to maximize the flow on the dummy arc.

These two issues imply that if we maximize the number of cars flowing on the dummy road $6 \rightarrow 1$, this is equivalent to maximizing the total number of cars flowing *out* of node 1 as well

FIGURE 5.7
Modified Road Network for Waukesha—Maximal-Flow



as the total number of cars flowing *in* to node 6. The objective for Waukesha's maximal-flow model can therefore be written as

$$\text{Maximize } X_{61}$$

CONSTRAINTS Because all nodes in the network are transshipment nodes with no supplies or demands, the flow balance equations need to ensure that the net flow (i.e., number of cars) at each node is zero. Hence,

All net flows are zero.

$$
\begin{array}{lll}
(X_{61} + X_{21}) - (X_{12} + X_{13} + X_{14}) & = 0 & \text{(net flow at node 1)} \\
(X_{12} + X_{42} + X_{62}) - (X_{21} + X_{24} + X_{26}) & = 0 & \text{(net flow at node 2)} \\
(X_{13} + X_{43} + X_{53}) - (X_{34} + X_{35}) & = 0 & \text{(net flow at node 3)} \\
(X_{14} + X_{24} + X_{34} + X_{64}) - (X_{42} + X_{43} + X_{46}) & = 0 & \text{(net flow at node 4)} \\
(X_{35}) - (X_{53} + X_{56}) & = 0 & \text{(net flow at node 5)} \\
(X_{26} + X_{46} + X_{56}) - (X_{61} + X_{62} + X_{64}) & = 0 & \text{(net flow at node 6)}
\end{array}
$$

Finally, we have capacity constraints on the maximum number of cars that can flow on each road. These are written as

Capacity constraints limit the flows on the arcs.

$$
\begin{array}{lll}
X_{12} \le 3 & X_{13} \le 10 & X_{14} \le 2 \\
X_{21} \le 1 & X_{24} \le 1 & X_{26} \le 2 \\
X_{34} \le 3 & X_{35} \le 2 & \\
X_{42} \le 1 & X_{43} \le 1 & X_{46} \le 1 \\
X_{53} \le 1 & X_{56} \le 6 & \\
X_{61} \le 1{,}000 & X_{62} \le 2 & X_{64} \le 1
\end{array}
$$

File: 5-7.xls

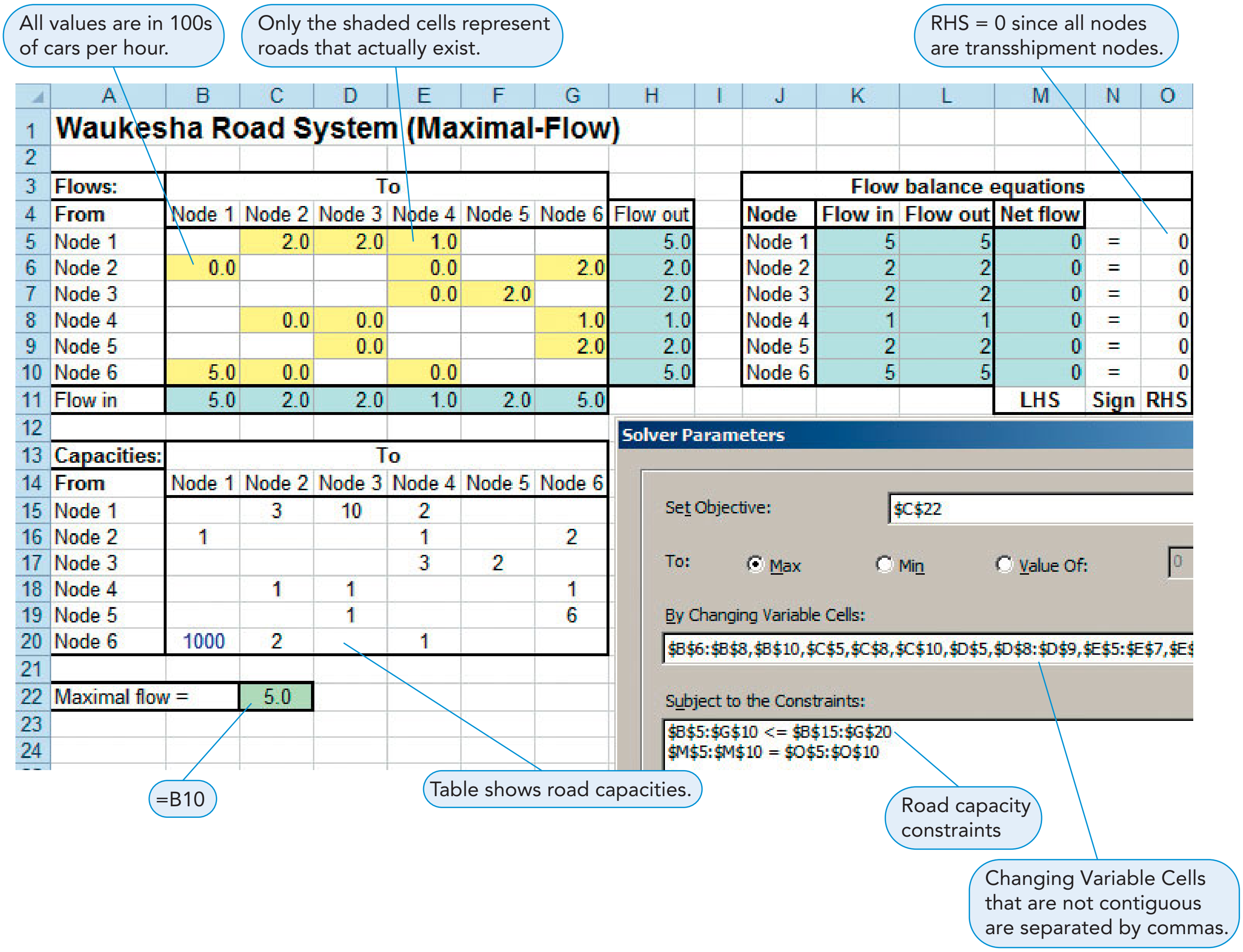
EXCEL SOLUTION Screenshot 5-7 shows the Excel layout and Solver entries for Waukesha's maximal-flow model. To be consistent with earlier models, flows on arcs have been modeled here using a tabular layout (cells B5:G10). As noted earlier, a big advantage of the tabular layout is that it greatly simplifies the calculations of the total flows in and total flows out of each node in the network.

However, of the 36 (= 6 × 6) arcs represented by cells B5:G10, only 16 of them actually exist in Waukesha's network. That is, the decision variables in this model refer only to selected entries in the table. These entries have been shaded yellow in Screenshot 5-7.

Entries for nonadjacent cells are separated by commas in Solver.

As with the Lopez Custom Outfits transshipment model we discussed in section 5.5, there are two ways of specifying the variables for a maximal-flow model in Solver. First, we could enter only the shaded cells, as illustrated in the By Changing Variable Cells box in Screenshot 5-7. Note that we separate entries for nonadjacent cells by using commas (i.e., B6:B8, B10, C5, etc.). As you can see, this approach could be cumbersome especially if there are many nonadjacent decision variables in the model.

SCREENSHOT 5-7 **Excel Layout and Solver Entries for Waukesha Road System—Maximal-Flow**



Flows: From	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Flow out
Node 1		2.0	2.0	1.0			5.0
Node 2	0.0			0.0		2.0	2.0
Node 3				0.0	2.0		2.0
Node 4		0.0	0.0			1.0	1.0
Node 5			0.0			2.0	2.0
Node 6	5.0	0.0		0.0			5.0
Flow in	5.0	2.0	2.0	1.0	2.0	5.0	

Node	Flow in	Flow out	Net flow (LHS)	Sign	RHS
Node 1	5	5	0	=	0
Node 2	2	2	0	=	0
Node 3	2	2	0	=	0
Node 4	1	1	0	=	0
Node 5	2	2	0	=	0
Node 6	5	5	0	=	0

Capacities: From	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6
Node 1		3	10	2		
Node 2	1			1		2
Node 3				3	2	
Node 4		1	1			1
Node 5			1			6
Node 6	1000	2		1		

Maximal flow = 5.0

Arc capacities of zero will prevent flows on arcs.

In the second (and easier) approach, we specify the entire cell range B5:G10 in the By Changing Variable Cells box in Solver.[4] Then, for all roads that do not exist (e.g., road 1 → 5, road 2 → 3), we set the flow capacity to zero. Solved Problem 5-3 at the end of this chapter shows an example of this approach for a maximal-flow model.

The solution to Waukesha's problem shows that 500 cars (recall that all numbers are in hundreds of cars) can flow through the town from west to east. The values of the decision variables indicate the actual car flow on each road. Total flow in (column K) and total flow out (column L) at each node are also shown. For example, the total flow out of node 1 is 500 cars, split as 200 cars on 1 → 2, 200 cars on 1 → 3, and 100 cars on 1 → 4.

5.8 Shortest-Path Model

A shortest-path model finds the path with the minimum distance through a network.

The *shortest-path model* finds how a person or an item can travel from one location to another through a network while minimizing the total distance traveled, time taken, or some other measure. In other words, it finds the shortest path or route from an origin to a series of destinations.

---

[4] If there are more than 14 nodes in the network, we cannot use this approach with the standard version of Solver (included with Excel) because that version can handle a maximum of only 200 decision variables.

IN ACTION Maximal-Flow Model Facilitates Improved Natural Gas Production and Transport

Norwegian gas covered three percent of the worldwide production in 2007, and its export is expected to increase by nearly 50 percent within the next decade. With over 7,800 km of subsea pipelines, the natural gas transport network on the Norwegian Continental Shelf (NCS) is the world's largest offshore pipeline network. The network has an annual capacity of 120 billion standard cubic meters, which represents about 15 percent of European consumption.

In order to ensure the most effective usage of this complex network, StatoilHydro, Norway's main shipper of natural gas, and Gassco, an independent network operator, use a maximum flow model embedded within a decision support tool named GassOpt to optimize the network configuration and routing. The primary decision variables in this model are the total flow and component flow between different nodes in the network. The objective function is to maximize the gas flow with some penalty terms for pressure increases, etc. The constraints include such issues as field capacities, market demands, mass balance, and pressure and flow restrictions.

GassOpt has been used extensively for the development of the dry-gas network on the NCS for the past decade, and is expected to remain an important part of infrastructure development. StatoilHydro estimates that its accumulated savings related to the use of GassOpt were approximately US$2 billion during the period 1995–2008.

Source: Based on F. Rømo et al. "Optimizing the Norwegian Natural Gas Production and Transport," *Interfaces* 39, 1 (January–February 2009): 46–56.

Ray Design Inc. Example

Every day, Ray Design Inc. must transport beds, chairs, and other furniture items from the factory to the warehouse. This involves going through several cities (nodes). Ray would like to find the path with the shortest distance, in miles. The road network is shown in Figure 5.8.

A shortest-path model has a unique starting node and a unique ending node.

The shortest-path model is another example of a network model that has a unique starting node (origin) and a unique ending node (destination). If we assume that there is a supply of one unit at node 1 (factory) and a demand of one unit at node 6 (warehouse), the shortest-path model for the Ray Design example is identical to a transshipment model with a single origin node (node 1), a single destination node (node 6), and four transshipment nodes (node 2 through node 5).

Each flow in a shortest-path model will equal one unit.

Because the supply and demand both equal one unit, which is a whole number, the solution to the model will have integer-valued flows on all arcs. Hence, the supply of one unit at node 1 will flow in its entirety on either road $1 \rightarrow 2$ or road $1 \rightarrow 3$. Further, because the net flow is zero at each of the transshipment nodes (cities), a flow of one unit on an incoming arc (road) at any of these cities automatically has to result in a flow of one unit on an outgoing road from that city.

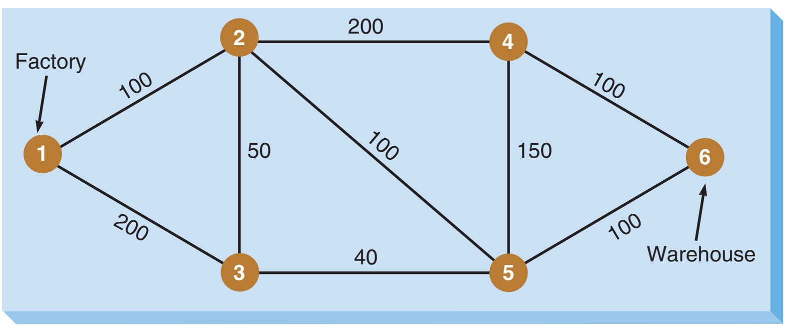
LP Formulation for Ray Design Inc.'s Shortest-Path Model

Because all 9 arcs (roads) in the network are bidirectional, we first replace each one with a pair of unidirectional roads. There are, therefore, 18 decision variables in the model. As usual, let

X_{ij} = Flow on road from node i to node j. The solution value will equal 1 if travel occurs on the road from node i to node j and will equal 0 otherwise.

where i and j each equal 1, 2, 3, 4, 5, or 6. As with the maximal-flow model, flow variables are defined only on roads that actually exist. For example, X_{12} (i.e., $i = 1, j = 2$) is defined, while X_{14} (i.e., $i = 1, j = 4$) is not defined.

FIGURE 5.8
Roads from Ray Design's Factory to Warehouse—Shortest-Path



OBJECTIVE FUNCTION The objective is to minimize the distance between node 1 and node 6 and can be expressed as

$$\begin{aligned}\text{Minimize } z\ & 100X_{12} + 200X_{13} + 100X_{21} + 50X_{23} + 200X_{24} + 100X_{25} \\ & + 200X_{31} + 50X_{32} + 40X_{35} + 200X_{42} + 150X_{45} + 100X_{46} \\ & + 100X_{52} + 40X_{53} + 150X_{54} + 100X_{56} + 100X_{64} + 100X_{65}\end{aligned}$$

The optimal value for each variable will be 0 or 1, depending on whether travel occurs on that road. So the objective function is the sum of road distances on which travel (flow) actually occurs.

CONSTRAINTS We write the flow balance constraints at each node as follows:

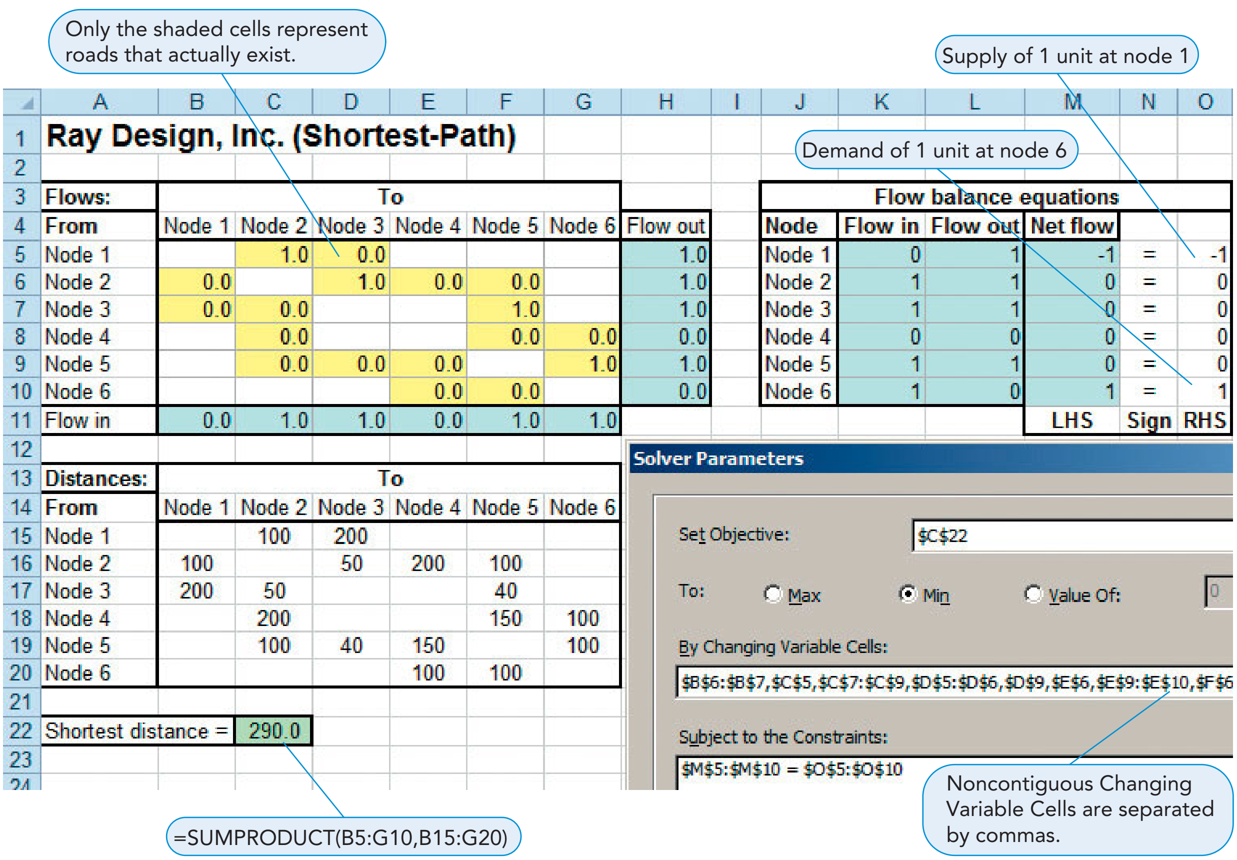
$$\begin{aligned}
(X_{21} + X_{31}) - (X_{12} + X_{13}) &= -1 && \text{(supply of one unit at node 1)} \\
(X_{12} + X_{32} + X_{42} + X_{52}) - (X_{21} + X_{23} + X_{24} + X_{25}) &= 0 && \text{(transshipment at node 2)} \\
(X_{13} + X_{23} + X_{53}) - (X_{31} + X_{32} + X_{35}) &= 0 && \text{(transshipment at node 3)} \\
(X_{24} + X_{54} + X_{64}) - (X_{42} + X_{45} + X_{46}) &= 0 && \text{(transshipment at node 4)} \\
(X_{25} + X_{35} + X_{45} + X_{65}) - (X_{52} + X_{53} + X_{54} + X_{56}) &= 0 && \text{(transshipment at node 5)} \\
(X_{46} + X_{56}) - (X_{64} + X_{65}) &= 1 && \text{(demand of one unit at node 6)}
\end{aligned}$$

WWW

File: 5-8.xls

EXCEL SOLUTION Screenshot 5-8 shows the Excel layout and Solver entries for Ray Design's shortest-path model. Once again, we use the tabular layout to represent flows on roads. However, only 18 of the 36 cells in the range B5:G10 actually represent roads that exist (indicated by the cells shaded yellow in Screenshot 5-8). As with the maximal-flow model, roads that do not exist need to be excluded when specifying entries for the changing variable cells in Solver.

SCREENSHOT 5-8 Excel Layout and Solver Entries for Ray Design Inc.—Shortest-Path



Ray Design, Inc. (Shortest-Path)

Flows:	To						
From	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Flow out
Node 1		1.0	0.0				1.0
Node 2	0.0		1.0	0.0	0.0		1.0
Node 3	0.0	0.0			1.0		1.0
Node 4		0.0			0.0	0.0	0.0
Node 5		0.0	0.0	0.0		1.0	1.0
Node 6				0.0	0.0		0.0
Flow in	0.0	1.0	1.0	0.0	1.0	1.0	

Flow balance equations					
Node	Flow in	Flow out	Net flow		
Node 1	0	1	-1	=	-1
Node 2	1	1	0	=	0
Node 3	1	1	0	=	0
Node 4	0	0	0	=	0
Node 5	1	1	0	=	0
Node 6	1	0	1	=	1
			LHS	Sign	RHS

Distances:	To					
From	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6
Node 1		100	200			
Node 2	100		50	200	100	
Node 3	200	50			40	
Node 4		200			150	100
Node 5		100	40	150		100
Node 6				100	100	

Shortest distance =	290.0

IN ACTION **Spanning Tree Analysis of a Telecommunications Network**

Network models have been used to solve a variety of problems of many different companies. In telecommunications, there is always a need to connect computer systems and devices together in an efficient and effective manner. Digital Equipment Corporation (DEC), for example, was concerned about how computer systems and devices were connected to a local area network (LAN) using Ethernet. The DEC net routing department was responsible for this and other network and telecommunications solutions.

Because of a number of technical difficulties, it was important to have an effective way to transport packets of information throughout the LAN. The solution was to use a spanning tree algorithm. The success of this approach can be seen in a poem written by one of the developers:

"I think I shall never see a graph more lovely than a tree.

A tree whose critical property is loop-free connectivity.

A tree that must be sure to span, so packets can reach every LAN.

First the route must be selected, by ID it is elected.

Least-cost paths from the root are traced.

In the tree these paths are placed.

A mesh is made for folks by me, then bridges find a spanning tree."

Source: Based on R. Perlman et al. "Spanning the LAN," Data Communications (October 21, 1997): 68.

One way of achieving this is to separate noncontiguous cell entries by using commas, as shown in Screenshot 5-8. That is, we specify only cells B6:B7, C5, C7:C9, etc. in the By Changing Variables Cells box in Solver.

Arcs with large (infinite) distances will have zero flows.

Alternatively, we can specify the entire cell range (cells B5:G10) as the changing variable cells.[5] However, to prevent travel on roads that do not exist (e.g., $1 \rightarrow 4$, $1 \rightarrow 5$), the distance of these roads can be set to a large number (compared to other distances in the problem) in the corresponding cells in B15:G20. For example, we could specify a large distance such as 2,000 for road $1 \rightarrow 4$ in cell E15. Clearly, no travel will occur on this road because the objective is to minimize total distance. Solved Problem 5-4 at the end of this chapter shows an example of this approach.

The solution to Ray's problem shows that the shortest distance from the factory to the warehouse is 290 miles and involves travel through cities 2, 3, and 5.

5.9 Minimal-Spanning Tree Model

A minimal-spanning tree model connects nodes at a minimum total distance.

The *minimal-spanning tree model* can be used to connect all the nodes of a network to each other while minimizing the total distance of all the arcs used for this connection. It has been applied, for example, by telephone companies to connect a number of phones (nodes) together while minimizing the total length of telephone cable (arcs).

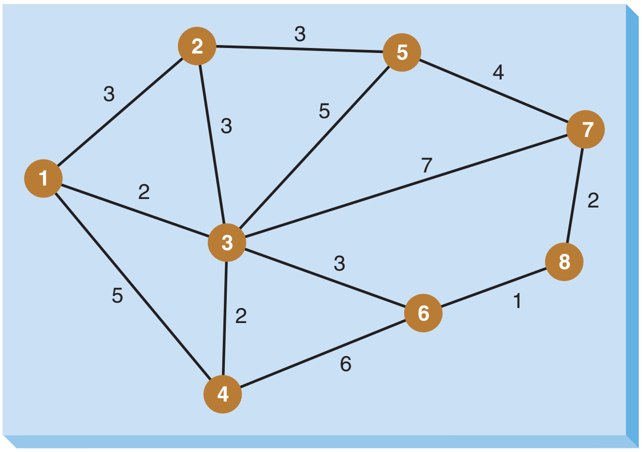
Lauderdale Construction Company Example

Let us consider the Lauderdale Construction Company, which is currently developing a luxurious housing project in Panama City, Florida. Melvin Lauderdale, owner of Lauderdale Construction, must determine the minimum total length of water pipes needed to provide water to each house. The network of eight houses is shown in Figure 5.9, along with the distances between the houses (in hundreds of feet).

Unlike the other network flow models studied so far in this chapter, in a minimal-spanning tree model, it is difficult to classify nodes *a priori* as origins, destinations, and transshipment nodes. For this reason, we do not formulate these models as LP problems using the flow balance

[5] As with the maximal-flow model, the standard version of Solver cannot handle this approach on networks with more than 14 nodes.

FIGURE 5.9
Network for Lauderdale Construction—Minimal-Spanning Tree



equations. However, the minimal-spanning tree model is very easy to solve by hand, using a simple solution procedure. The procedure is outlined as follows:

There are four steps in the solution procedure for minimal-spanning tree problems.

Steps for Solving the Minimal-Spanning Tree Model

1. Select any node in the network.
2. Connect this node to its nearest node.
3. Considering all the connected nodes, find the nearest *unconnected* node and then connect it. If there is a tie, and two or more unconnected nodes are equally near, select one arbitrarily. A tie suggests that there may be more than one optimal solution.
4. Repeat step 3 until all the nodes are connected.

Step 1: We select node 1.

Step 2: We connect node 1 to node 3.

Step 3: We connect the next nearest node (node 4).

Step 4: We repeat step 3 until all nodes are connected.

We can now solve the network in Figure 5.9 for Melvin Lauderdale. We start by arbitrarily selecting any node (house). Let's say we select house 1. Because house 3 is the nearest one to house 1, at a distance of 2 (200 feet), we connect these two houses. That is, we select arc $1 \longrightarrow 3$ for inclusion in the spanning tree. This is shown in Figure 5.10.

Next, considering connected houses 1 and 3, we look for the unconnected house that is closest to either house. This turns out to be house 4, which is 200 feet from house 3. We connect houses 3 and 4 by selecting arc $3 \longrightarrow 4$ (see Figure 5.11(a)).

We continue, looking for the nearest unconnected house to houses 1, 3, and 4. This is either house 2 or house 6, both at a distance of 300 feet from house 3. We arbitrarily pick house 2 and connect it to house 3 by selecting arc $3 \longrightarrow 2$ (see Figure 5.11(b)).

FIGURE 5.10
First Iteration for Lauderdale Construction

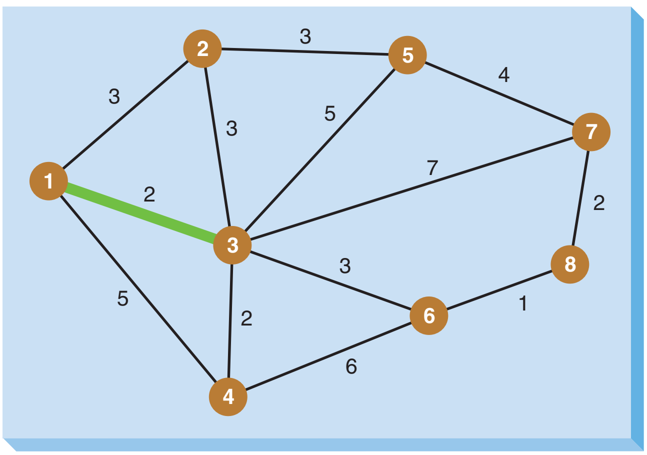
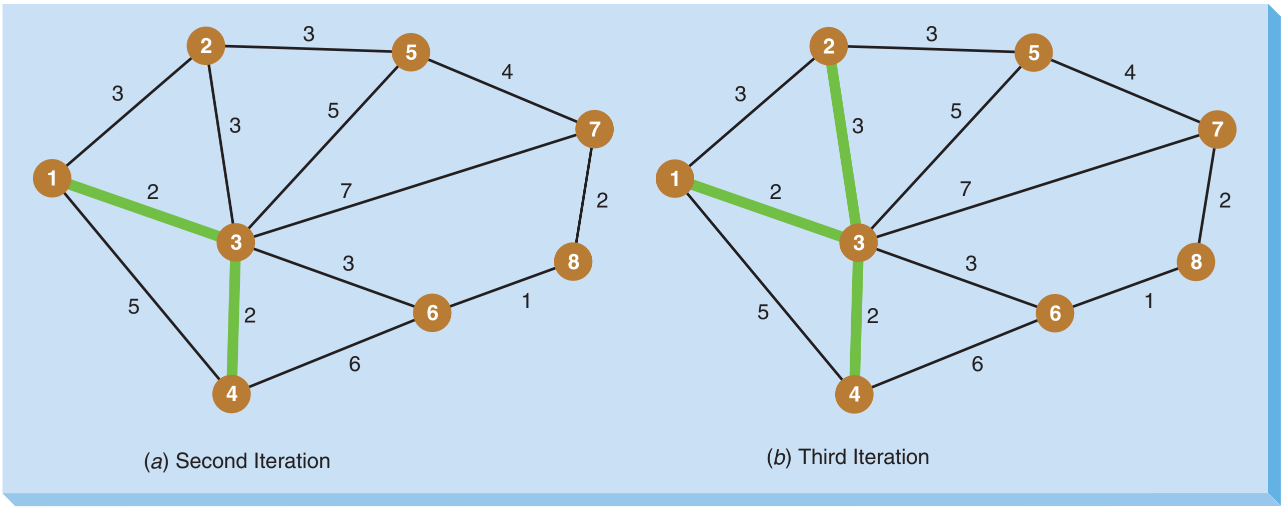


FIGURE 5.11 Second and Third Iterations for Lauderdale Construction



We continue the process. There is another tie for the next iteration, with a minimum distance of 300 feet (house 2 to house 5 and house 3 to house 6). Note that we do not consider house 1 to house 2, with a distance of 300 feet, at this iteration because both houses are already connected. We arbitrarily select house 5 and connect it to house 2 by selecting arc $2 \rightarrow 5$ (see Figure 5.12(a)). The next nearest house is house 6, and we connect it to house 3 by selecting arc $3 \rightarrow 6$ (see Figure 5.12(b)).

At this stage, we have only two unconnected houses left. House 8 is the nearest one to house 6, with a distance of 100 feet, and we connect it by using arc $6 \rightarrow 8$ (see Figure 5.13(a). Then the remaining house, house 7, is connected to house 8 using arc $8 \rightarrow 7$ (see Figure 5.13(b)).

Because there are no more unconnected houses, Figure 5.13(b) shows the final solution. Houses 1, 2, 4, and 6 are all connected to house 3. House 2 is connected to house 5. House 6 is connected to house 8, and house 8 is connected to house 7. The total distance is 1,600 feet.

FIGURE 5.12 Fourth and Fifth Iterations for Lauderdale Construction

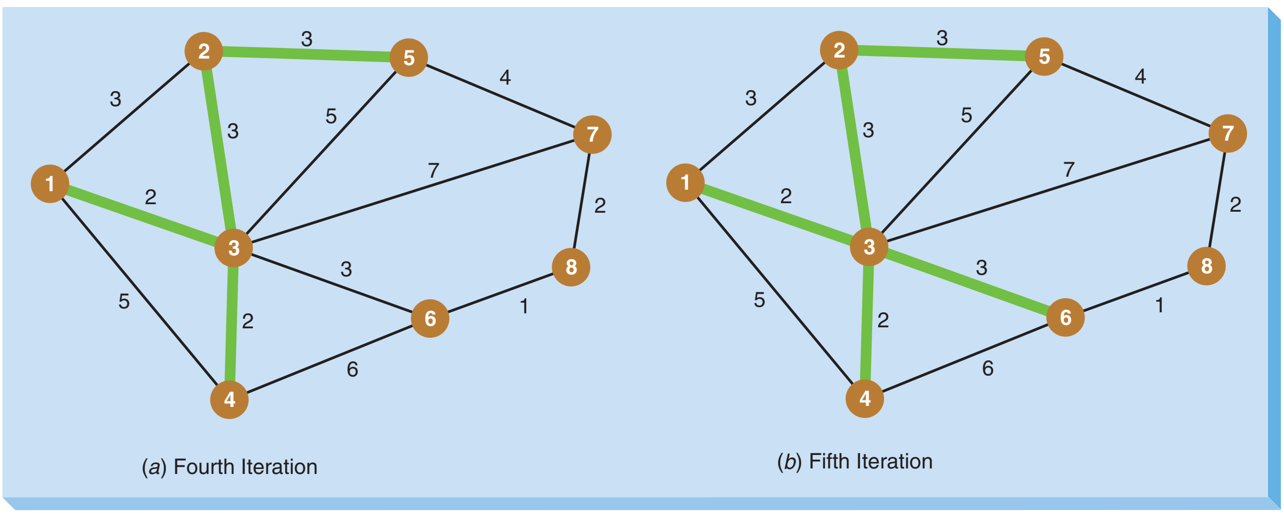
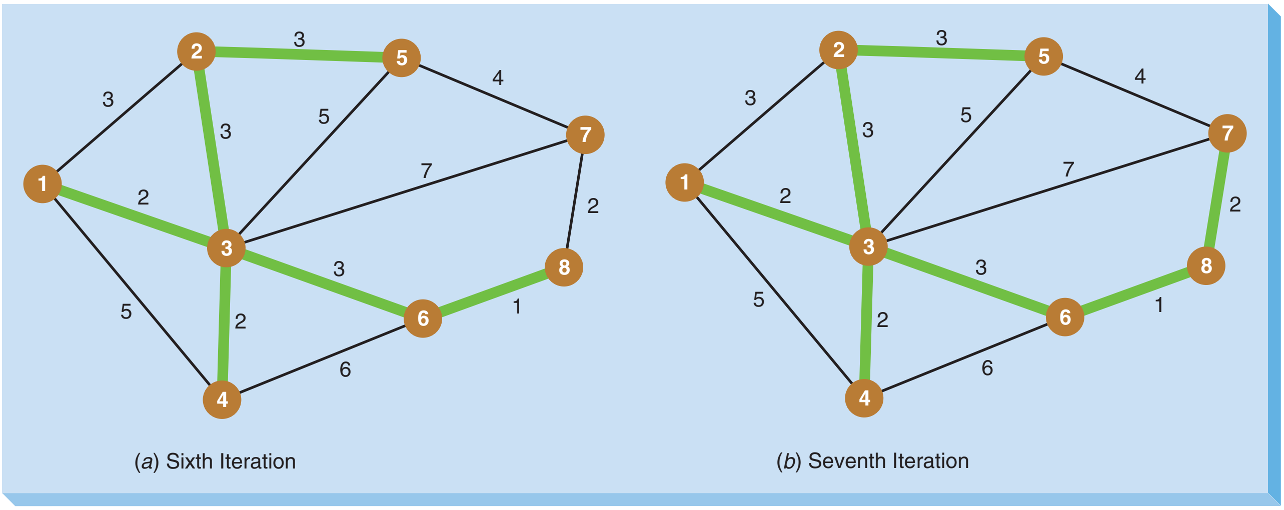


FIGURE 5.13 Sixth and Seventh (Final) Iterations for Lauderdale Construction



Summary

This chapter presents six important network flow models. First, we discuss the transportation model, which deals with the distribution of goods from several supply points to a number of demand points. We also consider transportation models with Max-Min and Min-Max objectives. We then extend the discussion to the transshipment model, which includes points that permit goods to both flow in and flow out of them. Next, we discuss the assignment model, which deals with determining the most efficient assignment of issues such as people to projects.

The fourth model covered is the maximal-flow model, which finds the maximum flow of any quantity or substance that can go through a network. This is followed by a discussion of the shortest-path model, which finds the shortest path through a network. Finally, we introduce the minimal-spanning tree model, which determines the path through the network that connects all the nodes while minimizing total distance.

Glossary

Assignment Model A specific type of network model that involves determining the most efficient assignment of people to projects, salespeople to territories, contracts to bidders, jobs to machines, and so on.

Balanced Model A model in which total demand (at all destinations) is equal to total supply (at all origins).

Destination A demand location in a transportation model. Also called a *sink*.

Facility Location Analysis An application of the transportation model to help a firm decide where to locate a new factory, warehouse, or other facility.

Max-Min Model A model that maximizes the minimum value of some or all of the decision variables.

Maximal-Flow Model A problem that finds the maximum flow of any quantity or substance through a network.

Min-Max Model A model that minimizes the maximum value of some or all of the decision variables.

Minimal-Spanning Tree Model A model that determines the path through the network that connects all the nodes while minimizing total distance.

Origin A supply location or source in a transportation model. Also called a *source*.

Shortest-Path Model A model that determines the shortest path or route through a network.

Transportation Model A specific network model case that involves scheduling shipments from origins to destinations so that total shipping costs are minimized.

Transshipment Model An extension of the transportation model in which some points have both flows in and out.

Unbalanced Model A situation in which total demand is not equal to total supply.

Solved Problems

Solved Problem 5-1

The Hardgrave Machine Company produces computer components at its factories in Cincinnati, Kansas City, and Pittsburgh. These factories have not been able to keep up with demand for orders at Hardgrave's four warehouses in Detroit, Houston, New York, and Los Angeles. As a result, the firm has decided to build a new factory to expand its productive capacity. The two sites being considered are Seattle, Washington, and Birmingham, Alabama. Both cities are attractive in terms of labor supply, municipal services, and ease of factory financing.

Table 5.6 presents the production costs and monthly supplies at each of the three existing factories, monthly demands at each of the four warehouses, and estimated production costs at the two proposed factories. Transportation costs from each factory to each warehouse are summarized in Table 5.7. Where should Hardgrave locate the new factory?

Solution

File: 5-9.xls

The total cost of each individual factory-to-warehouse route is found by adding the shipping costs (in the body of Table 5.7) to the respective unit production costs (from Table 5.6). For example, the total production plus shipping cost of one computer component from Cincinnati to Detroit is \$73 (= \$25 for shipping plus \$48 for production).

To determine which new factory (Seattle or Birmingham) yields the lowest total system cost, we solve two transportation models: one for each of the two possible locations. In each case, there are 4 factories and 4 warehouses. Hence, there are 16 decision variables.

Screenshots 5-9A and 5-9B on page 194 show the resulting optimum solutions with the total cost for each of the two locations. From these solutions, it appears that Seattle should be selected as the new factory site. Its total cost of \$3,704,000 is less than the \$3,741,000 cost at Birmingham.

TABLE 5.6
Hardgrave Machine's Demand and Supply Data

WAREHOUSE	MONTHLY DEMAND (UNITS)	PRODUCTION PLANT	MONTHLY SUPPLY	COST TO PRODUCE ONE UNIT
Detroit	10,000	Cincinnati	15,000	\$48
Houston	12,000	Kansas City	6,000	\$50
New York	15,000	Pittsburgh	14,000	\$52
Los Angeles	9,000		35,000	
	46,000			

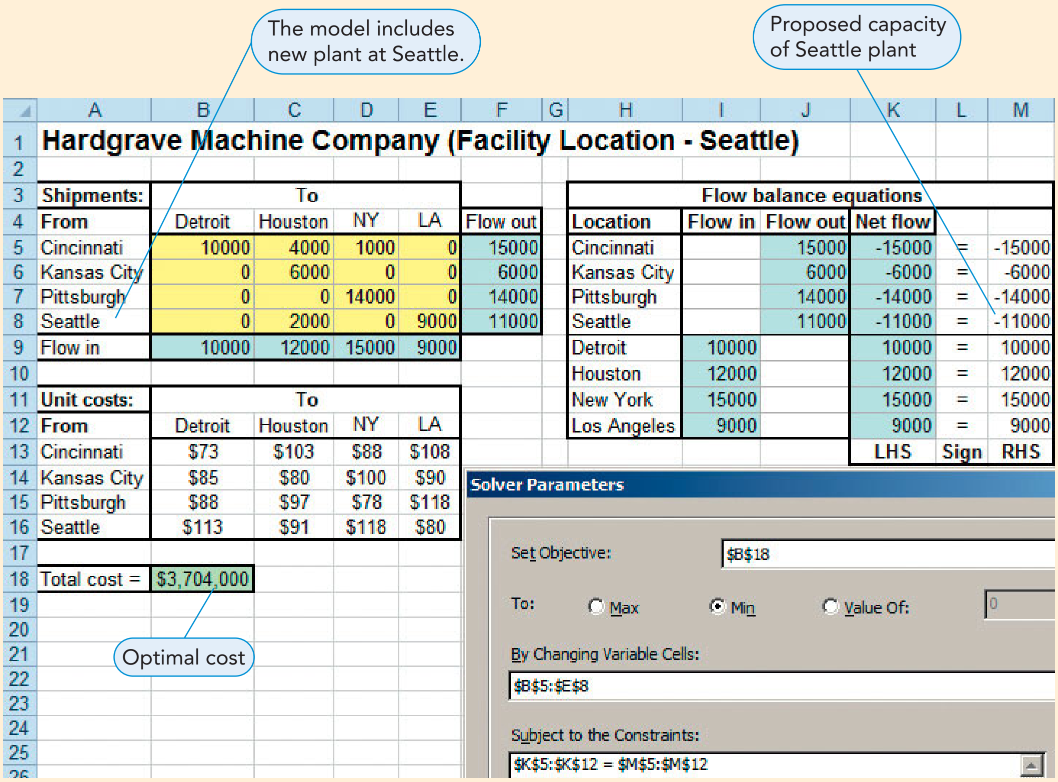
Supply needed from new plant $= 46{,}000 - 35{,}000 = 11{,}000$ units per month

ESTIMATED PRODUCTION COST PER UNIT AT PROPOSED PLANTS	
Seattle	\$53
Birmingham	\$49

TABLE 5.7
Hardgrave Machine's Shipping Costs

	TO			
FROM	DETROIT	HOUSTON	NEW YORK	LOS ANGELES
Cincinnati	\$25	\$55	\$40	\$60
Kansas City	\$35	\$30	\$50	\$40
Pittsburgh	\$36	\$45	\$26	\$66
Seattle	\$60	\$38	\$65	\$27
Birmingham	\$35	\$30	\$41	\$50

SCREENSHOT 5-9A
Excel Layout and Solver Entries for Hardgrave Machine—New Facility in Seattle



Hardgrave Machine Company (Facility Location - Seattle)

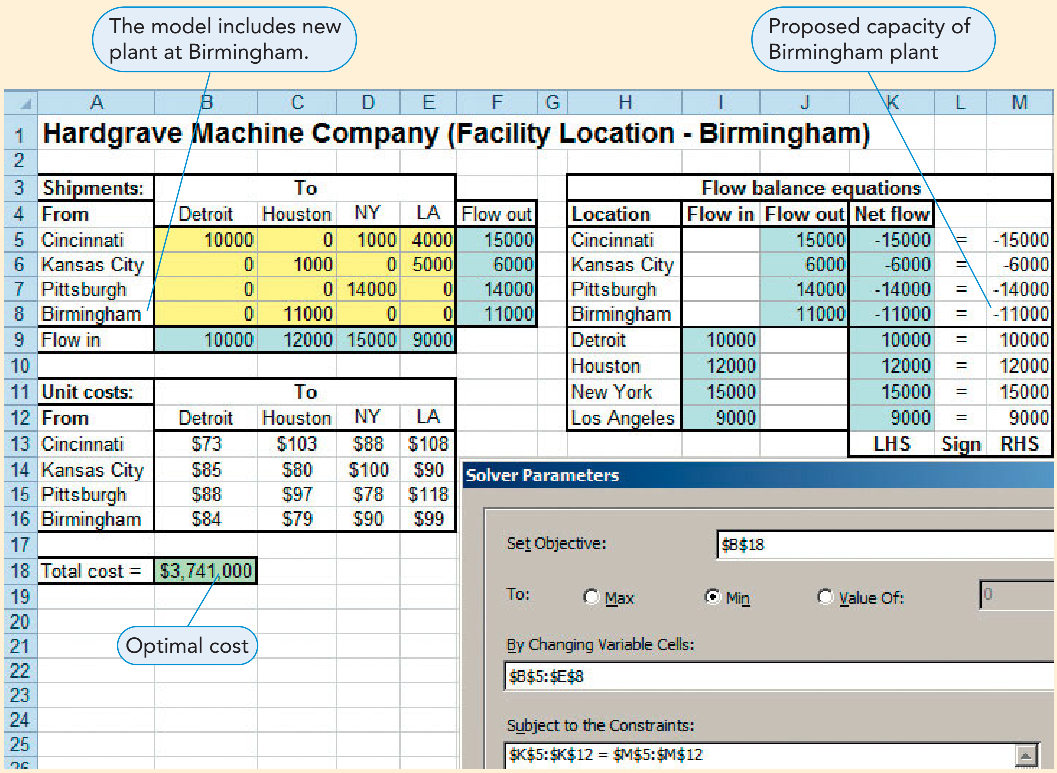
Shipments:	To				
From	Detroit	Houston	NY	LA	Flow out
Cincinnati	10000	4000	1000	0	15000
Kansas City	0	6000	0	0	6000
Pittsburgh	0	0	14000	0	14000
Seattle	0	2000	0	9000	11000
Flow in	10000	12000	15000	9000	

Unit costs:	To			
From	Detroit	Houston	NY	LA
Cincinnati	$73	$103	$88	$108
Kansas City	$85	$80	$100	$90
Pittsburgh	$88	$97	$78	$118
Seattle	$113	$91	$118	$80

Total cost = $3,704,000

Flow balance equations					
Location	Flow in	Flow out	Net flow		
Cincinnati		15000	-15000	=	-15000
Kansas City		6000	-6000	=	-6000
Pittsburgh		14000	-14000	=	-14000
Seattle		11000	-11000	=	-11000
Detroit	10000		10000	=	10000
Houston	12000		12000	=	12000
New York	15000		15000	=	15000
Los Angeles	9000		9000	=	9000
			LHS	Sign	RHS

SCREENSHOT 5-9B
Excel Layout and Solver Entries for Hardgrave Machine—New Facility in Birmingham



Hardgrave Machine Company (Facility Location - Birmingham)

Shipments:	To				
From	Detroit	Houston	NY	LA	Flow out
Cincinnati	10000	0	1000	4000	15000
Kansas City	0	1000	0	5000	6000
Pittsburgh	0	0	14000	0	14000
Birmingham	0	11000	0	0	11000
Flow in	10000	12000	15000	9000	

Unit costs:	To			
From	Detroit	Houston	NY	LA
Cincinnati	$73	$103	$88	$108
Kansas City	$85	$80	$100	$90
Pittsburgh	$88	$97	$78	$118
Birmingham	$84	$79	$90	$99

Total cost = $3,741,000

Flow balance equations					
Location	Flow in	Flow out	Net flow		
Cincinnati		15000	-15000	=	-15000
Kansas City		6000	-6000	=	-6000
Pittsburgh		14000	-14000	=	-14000
Birmingham		11000	-11000	=	-11000
Detroit	10000		10000	=	10000
Houston	12000		12000	=	12000
New York	15000		15000	=	15000
Los Angeles	9000		9000	=	9000
			LHS	Sign	RHS

Solved Problem 5-2

Prentice Hall Inc., a publisher headquartered in New Jersey, wants to assign three recently hired college graduates, Jones, Smith, and Wilson, to regional sales offices in Omaha, Miami, and Dallas. But the firm also has an opening in New York and would send one of the three there if it were more economical than a move to Omaha, Miami, or Dallas. It will cost $1,000 to relocate Jones to New York, $800 to relocate Smith there, and $1,500 to move Wilson. The other relocation costs are as follows:

	OFFICE		
HIREE	OMAHA	MIAMI	DALLAS
Jones	$800	$1,100	$1,200
Smith	$500	$1,600	$1,300
Wilson	$500	$1,000	$2,300

What is the optimal assignment of personnel to offices?

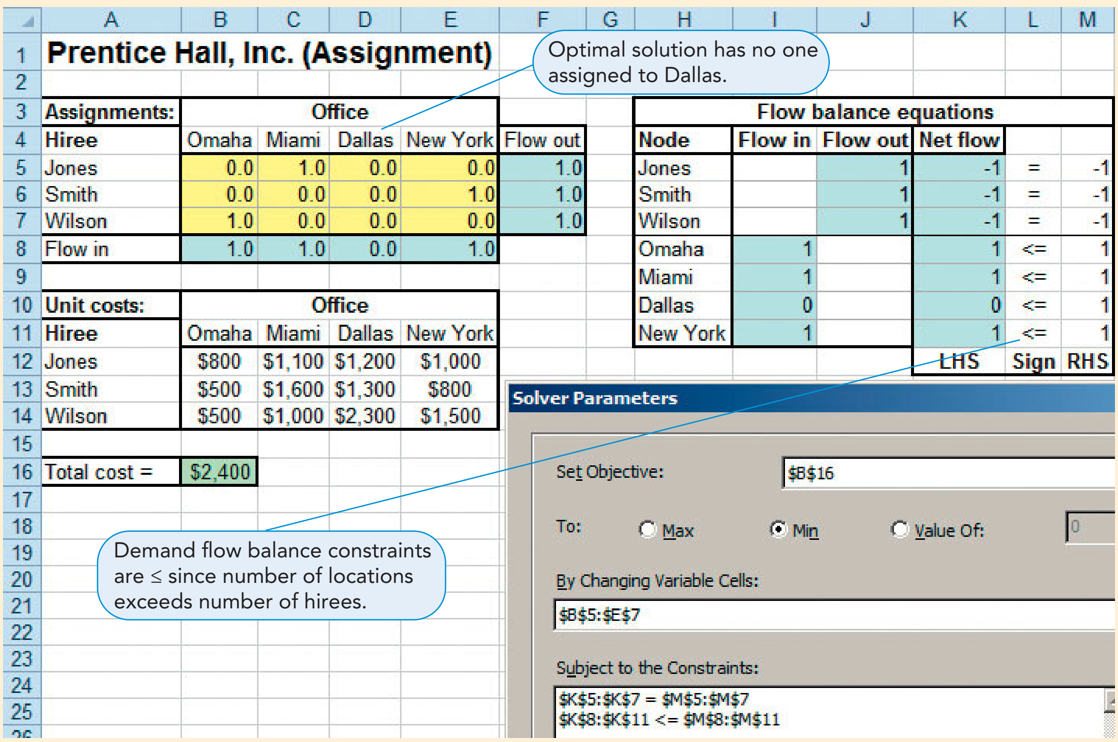
File: 5-10.xls

Solution

Because this is an unbalanced assignment model with three supply points (hirees) and four demand points (offices), note that the demand constraints should be expressed as inequalities (i.e., they should have ≤ signs).

Screenshot 5-10 shows the Excel layout and solution for Prentice Hall's assignment model. The optimal solution is to assign Wilson to Omaha, Smith to New York, and Jones to Miami. Nobody is assigned to Dallas. The total cost is $2,400.

SCREENSHOT 5-10 Excel Layout and Solver Entries for Prentice Hall Inc.—Assignment



Solved Problem 5-3

PetroChem, an oil refinery located on the Mississippi River south of Baton Rouge, Louisiana, is designing a new plant to produce diesel fuel. Figure 5.14 shows the network of the main processing centers along with the existing rate of flow (in thousands of gallons of fuel). The management at PetroChem would like to determine the maximum amount of fuel that can flow through the plant, from node 1 to node 7.

Solution

Node 1 is the origin node, and node 7 is the destination node. As described in Section 5.7, we convert all bidirectional arcs into unidirectional arcs, and we introduce a dummy arc from node 7 to node 1. The modified network is shown in Figure 5.15. The capacity of the dummy arc from node 7 to node 1 is set at a large number such as 1,000.

Screenshot 5-11 on page 197 shows the Excel layout and solution for this model. Unlike our earlier maximal-flow example in Screenshot 5-7, the entire cell range B5:H11 has been specified as the changing variable cells in Solver. However, the capacity of all arcs that do not exist (shown by the non-yellow cells in B5:H11) has been set to zero (in cells B16:H22) to prevent any fuel flows on these pipes (arcs).

The optimal solution shows that it is possible to have 10,000 gallons flow from node 1 to node 7 using the existing network.

Solved Problem 5-4

The network in Figure 5.16 shows the roads and cities surrounding Leadville, Colorado. Leadville Tom, a bicycle helmet manufacturer, must transport his helmets to a distributor based in Dillon, Colorado. To do this, he must go through several cities. Tom would like to find the shortest way to get from Leadville to Dillon. What do you recommend?

FIGURE 5.14
Network for PetroChem—Maximal-Flow

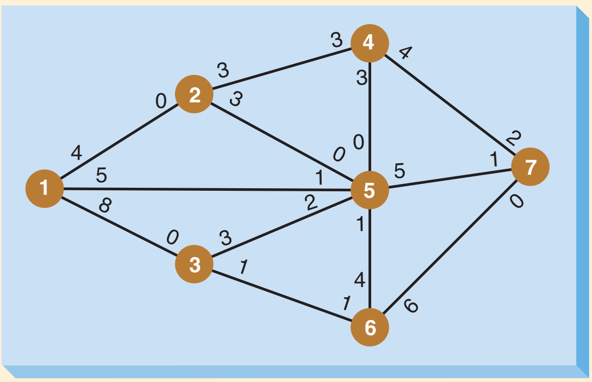
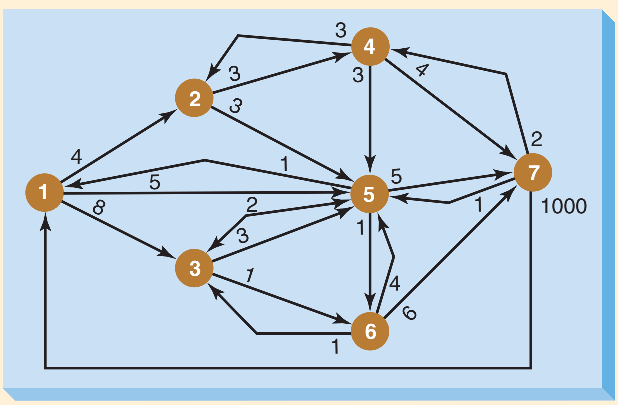
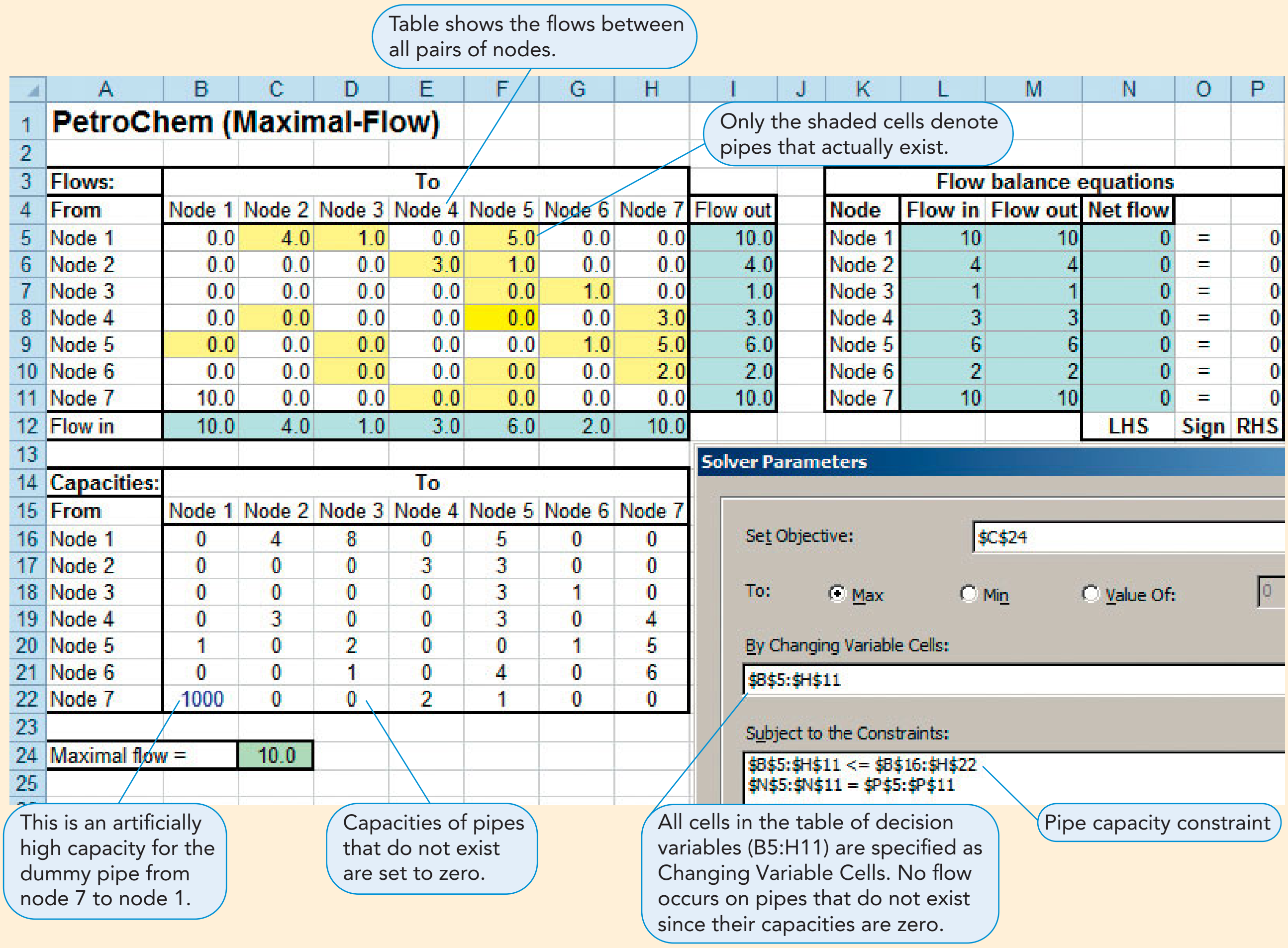


FIGURE 5.15
Modified Network for PetroChem—Maximal-Flow



SCREENSHOT 5-11 Excel Layout and Solver Entries for PetroChem—Maximal-Flow



PetroChem (Maximal-Flow)

Flows:	To							
From	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7	Flow out
Node 1	0.0	4.0	1.0	0.0	5.0	0.0	0.0	10.0
Node 2	0.0	0.0	0.0	3.0	1.0	0.0	0.0	4.0
Node 3	0.0	0.0	0.0	0.0	0.0	1.0	0.0	1.0
Node 4	0.0	0.0	0.0	0.0	0.0	0.0	3.0	3.0
Node 5	0.0	0.0	0.0	0.0	0.0	1.0	5.0	6.0
Node 6	0.0	0.0	0.0	0.0	0.0	0.0	2.0	2.0
Node 7	10.0	0.0	0.0	0.0	0.0	0.0	0.0	10.0
Flow in	10.0	4.0	1.0	3.0	6.0	2.0	10.0	

Flow balance equations					
Node	Flow in	Flow out	Net flow		
Node 1	10	10	0	=	0
Node 2	4	4	0	=	0
Node 3	1	1	0	=	0
Node 4	3	3	0	=	0
Node 5	6	6	0	=	0
Node 6	2	2	0	=	0
Node 7	10	10	0	=	0
			LHS	Sign	RHS

Capacities:	To						
From	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7
Node 1	0	4	8	0	5	0	0
Node 2	0	0	0	3	3	0	0
Node 3	0	0	0	0	3	1	0
Node 4	0	3	0	0	3	0	4
Node 5	1	0	2	0	0	1	5
Node 6	0	0	1	0	4	0	6
Node 7	1000	0	0	2	1	0	0

Maximal flow = 10.0

Solver Parameters

Set Objective: $C$24

To: Max Min Value Of: 0

By Changing Variable Cells: $B$5:$H$11

Subject to the Constraints:
$B$5:$H$11 <= $B$16:$H$22
$N$5:$N$11 = $P$5:$P$11

Table shows the flows between all pairs of nodes.

Only the shaded cells denote pipes that actually exist.

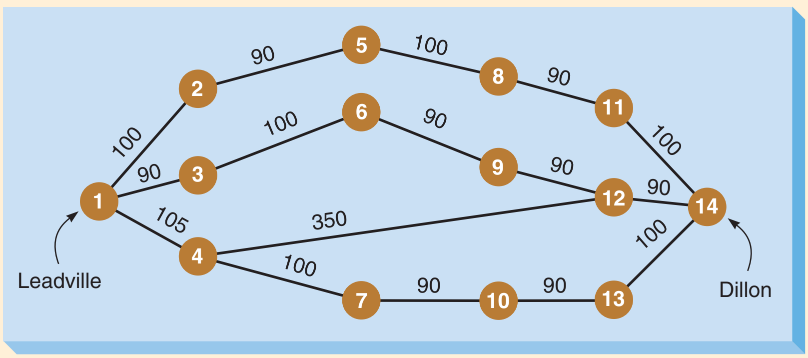
This is an artificially high capacity for the dummy pipe from node 7 to node 1.

Capacities of pipes that do not exist are set to zero.

All cells in the table of decision variables (B5:H11) are specified as Changing Variable Cells. No flow occurs on pipes that do not exist since their capacities are zero.

Pipe capacity constraint

FIGURE 5.16
Network for Leadville Tom—Shortest-Path



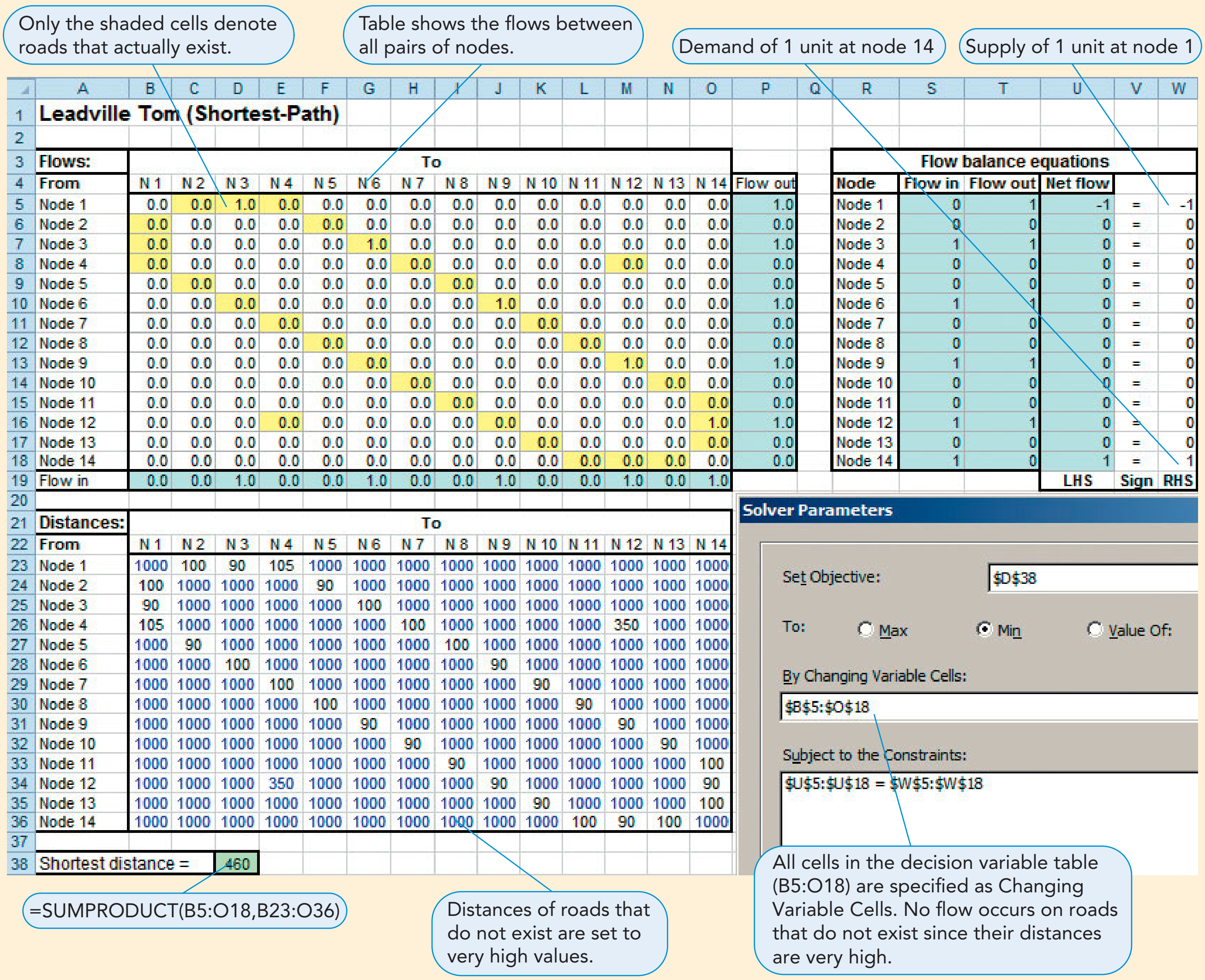
Solution

We associate a supply of one unit at Leadville (node 1) and a demand of one unit at Dillon (node 14). Screenshot 5-12 shows the Excel layout and solution for this model. Unlike our earlier shortest-path example in Screenshot 5-8, the entire cell range B5:O18 has been specified as the changing variable cells in Solver. However, we prevent flow (travel) on all roads that do not exist (shown by the non-yellow cells in B5:O18) by setting their distances to high values (in cells B23:O36).

The optimal solution shows that the shortest distance from Leadville to Dillon is 460 miles and involves travel through nodes 3, 6, 9, and 12.

WWW
File: 5-11.xls

SCREENSHOT 5-12 **Excel Layout and Solver Entries for Leadville Tom—Shortest-Path**



Leadville Tom (Shortest-Path)

Flows: From	N 1	N 2	N 3	N 4	N 5	N 6	N 7	N 8	N 9	N 10	N 11	N 12	N 13	N 14	Flow out
Node 1	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0
Node 2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Node 3	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0
Node 4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Node 5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Node 6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0
Node 7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Node 8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Node 9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	1.0
Node 10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Node 11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Node 12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	1.0
Node 13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Node 14	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Flow in	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	1.0	

Flow balance equations

Node	Flow in	Flow out	Net flow		
Node 1	0	1	-1	=	-1
Node 2	0	0	0	=	0
Node 3	1	1	0	=	0
Node 4	0	0	0	=	0
Node 5	0	0	0	=	0
Node 6	1	1	0	=	0
Node 7	0	0	0	=	0
Node 8	0	0	0	=	0
Node 9	1	1	0	=	0
Node 10	0	0	0	=	0
Node 11	0	0	0	=	0
Node 12	1	1	0	=	0
Node 13	0	0	0	=	0
Node 14	1	0	1	=	1
			LHS	Sign	RHS

Distances: From	N 1	N 2	N 3	N 4	N 5	N 6	N 7	N 8	N 9	N 10	N 11	N 12	N 13	N 14
Node 1	1000	100	90	105	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
Node 2	100	1000	1000	1000	90	1000	1000	1000	1000	1000	1000	1000	1000	1000
Node 3	90	1000	1000	1000	1000	100	1000	1000	1000	1000	1000	1000	1000	1000
Node 4	105	1000	1000	1000	1000	1000	100	1000	1000	1000	1000	350	1000	1000
Node 5	1000	90	1000	1000	1000	1000	1000	100	1000	1000	1000	1000	1000	1000
Node 6	1000	1000	100	1000	1000	1000	1000	1000	90	1000	1000	1000	1000	1000
Node 7	1000	1000	1000	100	1000	1000	1000	1000	1000	90	1000	1000	1000	1000
Node 8	1000	1000	1000	1000	100	1000	1000	1000	1000	1000	90	1000	1000	1000
Node 9	1000	1000	1000	1000	1000	90	1000	1000	1000	1000	1000	90	1000	1000
Node 10	1000	1000	1000	1000	1000	1000	90	1000	1000	1000	1000	1000	90	1000
Node 11	1000	1000	1000	1000	1000	1000	1000	90	1000	1000	1000	1000	1000	100
Node 12	1000	1000	1000	350	1000	1000	1000	1000	90	1000	1000	1000	1000	90
Node 13	1000	1000	1000	1000	1000	1000	1000	1000	1000	90	1000	1000	1000	100
Node 14	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	100	90	100	1000

Shortest distance = 460

Solved Problem 5-5

Roxie LaMothe, owner of a large horse breeding farm near Orlando, is planning to install a complete water system connecting all the various stables and barns. The locations of the facilities and the distances between them are given in the network shown in Figure 5.17. Roxie must determine the least expensive way to provide water to each facility. What do you recommend?

FIGURE 5.17
Network for Roxie LaMothe—Minimal-Spanning Tree

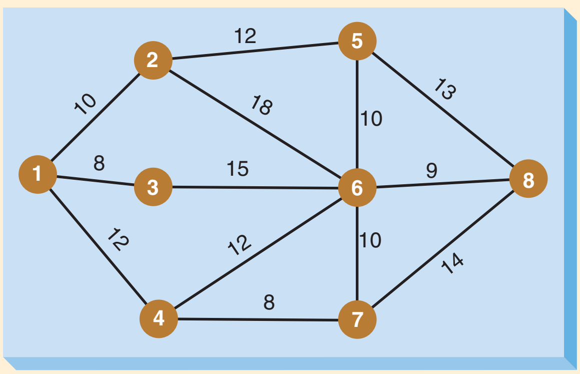
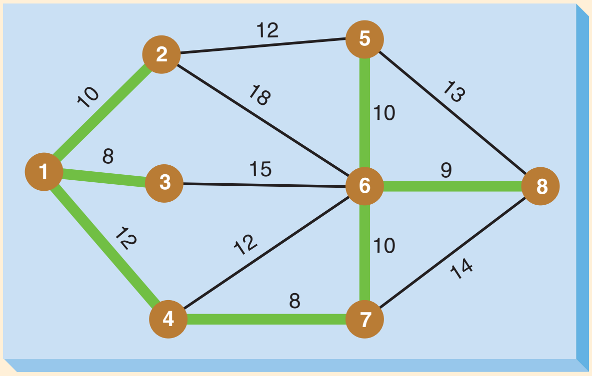


FIGURE 5.18
Minimal-Spanning Tree for Roxie LaMothe



Solution

This is a typical minimal-spanning tree problem that can be solved by hand. We begin by selecting node 1 and connecting it to the nearest node, which is node 3. Nodes 1 and 2 are the next to be connected, followed by nodes 1 and 4. Now we connect node 4 to node 7 and node 7 to node 6. At this point, the only remaining points to be connected are node 6 to node 8 and node 6 to node 5. The final solution is shown in Figure 5.18.

Discussion Questions and Problems

Discussion Questions

5-1 Is the transportation model an example of decision making under certainty or decision making under uncertainty? Why?

5-2 What is a balanced transportation model? Describe the approach you would use to solve an unbalanced model.

5-3 What is the enumeration approach to solving assignment models? Is it a practical way to solve a 5 row $\times$ 5 column model? a 7 $\times$ 7 model? Why?

5-4 What is the minimal-spanning tree model? What types of problems can be solved using this type of model?

5-5 Give several examples of problems that can be solved using the maximal-flow model.

5-6 Describe a problem that can be solved by using the shortest-path model.

5-7 What is a flow balance constraint? How is it implemented at each node in a network model?

5-8 How can we manipulate a maximal-flow network model in order to set it up as a linear program?

5-9 Why might it be more convenient to set up network models in Excel by using a tabular form?

5-10 How can we manipulate a maximal-flow network model in order to specify all arcs between each pair of nodes (i.e., the entire table) as the changing variable cells in Solver?

5-11 How can we manipulate a shortest-path network model in order to specify all arcs between each pair of nodes (i.e., the entire table) as the changing variable cells in Solver?

Problems

Note: The networks for all problems given here involve no more than 14 nodes. If we arrange the decision variables in tabular form, the total number of entries will be no more than 196 (= 14 $\times$ 14). Therefore, it should be possible to specify the entire table as changing variable cells even in the standard version of Solver.

5-12 The Oconee County, South Carolina, superintendent of education is responsible for assigning students to the three high schools in his county. A certain number of students have to travel to and from school by bus, as several sectors of the county are beyond walking distance from a school. The superintendent partitions the county into five geographic sectors as he attempts to establish a plan that will minimize the total number of student miles traveled by bus. Of course, if a student happens to live in a certain sector and is assigned to the high school in that sector, there is no need to bus that student because he or she can walk to school. The three schools are located in sectors B, C, and E. The table at the top of the next page reflects the number of high-school-age students living in each sector and the distance, in miles, from each sector to each school. Assuming that each high school has a capacity of 1,100 students, set up and solve Oconee County's problem as a transportation model.

SECTOR	DISTANCE TO SCHOOLS, IN MILES SECTOR B	SECTOR C	SECTOR E	NUMBER OF STUDENTS
A	6	7	11	800
B	0	3	10	600
C	9	0	6	400
D	8	3	5	700
E	15	8	0	500

5-13 Marc Hernandez's construction firm currently has three projects in progress. Each requires a specific supply of gravel. There are three gravel pits available to provide for Hernandez's needs, but shipping costs differ from location to location. The following table summarizes the transportation costs:

FROM	TO JOB 1	JOB 2	JOB 3	TONNAGE ALLOWANCE
Central pit	$9	$ 8	$ 7	3,000
Rock pit	$7	$11	$ 6	4,000
Acme pit	$4	$ 3	$12	6,000
Job requirements (tons)	2,500	3,750	4,850	

(a) Determine Hernandez's optimal shipping quantities so as to minimize total transportation costs.
(b) It is the case that Rock Pit and Central Pit can send gravel by rail to Acme for $1 per ton. Once the gravel is relocated, it can be trucked to the jobs. Reformulate this problem to determine how shipping by rail could reduce the transportation costs for the gravel.

5-14 The Southern Rail Company ships coal by rail from three coal mines to meet the demand requirements of four coal depots. The following table shows the distances from the mines to the various depots and the availabilities and requirements for coal. Determine the best shipment of coal cars to minimize the total miles traveled by the cars.

5-15 The Piedmont Investment Corporation has identified four small apartment buildings in which it would like to invest. The four banks generally used by Piedmont have provided quotes on the interest rates they would charge to finance each purchase. The banks have also advised Piedmont of the maximum amount of money they are willing to lend at this time. Piedmont would like to purchase as many buildings as possible while paying the lowest possible amount in total interest. More than one bank can be used to finance the same property. What should Piedmont do?

SAVINGS AND LOAN COMPANY	PROPERTY (INTEREST RATES) HILL ST.	BANKS ST.	PARK AVE.	DRURY LANE	MAX CREDIT LINE
First Homestead	8%	8%	10%	11%	$80,000
Commonwealth	9%	9%	12%	10%	$100,000
Washington Federal	9%	11%	10%	9%	$120,000
Loan required	$60,000	$40,000	$130,000	$70,000	

5-16 The manager of the O'Brian Glass Company is planning the production of automobile windshields for the next four months. The demand for the next four months is projected to be as shown in the following table.

MONTH	DEMAND FOR WINDSHIELDS
1	130
2	140
3	260
4	120

O'Brian can normally produce 100 windshields in a month. This is done during regular production hours at a cost of $100 per windshield. If demand in any one month cannot be satisfied by regular production, the production manager has three other choices: (1) He can produce up to 50 more windshields per month in

Table for Problem 5-14

FROM	TO COLUMBIA	ALBANY	SPRINGFIELD	PLEASATANBURG	SUPPLY OF CARS
Parris	50	30	60	70	35
Butler	20	80	10	90	60
Century	100	40	80	30	25
Demand for cars	30	45	25	20	

overtime but at a cost of $130 per windshield; (2) he can purchase a limited number of windshields from a friendly competitor for resale at a cost of $150 each (the maximum number of outside purchases over the four-month period is 450 windshields); or (3) he can fill the demand from his on-hand inventory. The inventory carrying cost is $10 per windshield per month. Back orders are not permitted. Inventory on hand at the beginning of month 1 is 40 windshields. Set up and solve this "production smoothing" problem as a transportation model to minimize cost. *Hint:* Set the various production options (e.g., regular production, outside purchase, etc.) as supply nodes and the monthly demands as the demand nodes.

5-17 Maurice's Pump Manufacturing Company currently maintains plants in Atlanta and Tulsa that supply major distribution centers in Los Angeles and New York. Because of an expanding demand, Maurice has decided to open a third plant and has narrowed the choice to one of two cities—New Orleans or Houston. The pertinent production and distribution costs, as well as the plant capacities and distribution center demands, are shown in the following table.

	DISTRIBUTION CENTERS			
PLANTS	LOS ANGELES	NEW YORK	CAPACITY	PRODUCTION COST (PER UNIT)
Atlanta (existing)	$8	$5	600	$6
Tulsa (existing)	$4	$7	900	$5
New Orleans (proposed)	$5	$6	500	$4 (*anticipated*)
Houston (proposed)	$4	$6	500	$3 (*anticipated*)
Forecast demand	800	1,200		

Which of the new possible plants should be opened?

5-18 A food distribution company ships fresh spinach from its four packing plants to large East-coast cities. The shipping costs per crate, the supply and demand are shown in the table at the bottom of this page.

(a) Formulate a model that will permit the company to meet its demand at the lowest possible cost.

(b) The company wishes to spread out the source for each of its markets to the maximum extent possible. To accomplish this, it will accept a 5% increase in its total transportation cost from part (a). What is the new transportation plan, and what is the new cost?

5-19 The Lilly Snack Company is considering adding an additional plant to its three existing facilities in Wise, Virginia; Humbolt, Tennessee; and Cleveland, Georgia to serve three large markets in the Southeast. Two locations—Brevard, North Carolina, and Laurens, South Carolina—are being considered. The transportation costs per pallet are shown in the table at the top of the next page.

(a) Which site would you recommend? Why?

(b) Suppose that the Brevard location has been selected. Due to the perishable nature of the goods involved, management wishes to restrict the maximum number of pallets shipped from any one plant to any single market. To accomplish this, management is willing to accept a 10% surcharge on their optimal transportation costs from part (a). What is the new transportation plan, and what is the new cost?

5-20 Meg Bishop, vice president of supply chain at the Lilly Snack Company (see Problem 5-19) has been able to secure shipping from the proposed plant in Brevard to its plants in Wise and Cleveland for $6 and $5 per pallet, respectively. If Lilly chooses to place a new plant in Brevard, what would be the new shipping plan and cost? Ignore part (b) of Problem 5-19 in answering this question.

5-21 The distribution system for the Smith Company consists of three plants (A, B, and C), two warehouses (D and E), and four customers (W, X, Y, and Z). The relevant supply, demand, and unit shipping cost information are given in the table for Problem 5-21 near the top of the next page. Set up and solve the transshipment model to minimize total shipping costs.

Table for Problem 5-18

	MARKETS				
PACKING PLANTS	ATLANTA	BOSTON	CHARLESTOWN	DOVER	SUPPLY
Eaglestown	$6.00	$7.00	$7.50	$7.50	8,000
Farrier	$5.50	$5.50	$4.00	$7.00	10,000
Guyton	$6.00	$5.00	$6.50	$7.00	5,000
Hayesville	$7.00	$7.50	$8.50	$6.50	9,000
Demand	8,000	9,000	10,000	5,000	

Table for Problem 5-19

	FROM					
TO	WISE	HUMBOLT	CLEVELAND	BREVARD	LAURENS	DEMAND
Charlotte	$20	$17	$21	$29	$27	250
Greenville	$25	$27	$20	$30	$28	200
Atlanta	$22	$25	$22	$30	$31	350
Capacity	300	200	150	150	150	

Table for Problem 5-21

PLANT	SUPPLY
A	450
B	500
C	380

CUSTOMER	DEMAND
W	450
X	300
Y	300
Z	400

	TO	
FROM	D	E
A	$4	$7
B	$8	$5
C	$5	$6

	TO			
FROM	W	X	Y	Z
D	$6	$4	$8	$4
E	$3	$6	$7	$7

5-22 A supply chain consists of three plants (A, B, and C), three distributors (J, K, and L), and three stores (X, Y, and Z). The relevant supply, demand, and unit shipping cost information are given in the table at the bottom of this page. Set up and solve the transshipment model to minimize total shipping costs.

5-23 In a job shop operation, four jobs can be performed on any of four machines. The hours required for each job on each machine are presented in the following table:

	MACHINE			
JOB	W	X	Y	Z
A	16	14	10	13
B	15	13	12	12
C	12	12	9	11
D	18	16	14	16

The plant supervisor would like to assign jobs so that total time is minimized. Use the assignment model to find the best solution.

5-24 Greg Pickett, coach of a little-league baseball team, is preparing for a series of four games against four good opponents. Greg would like to increase the probability of winning as many games as possible by carefully scheduling his pitchers against the teams they are each most likely to defeat. Because the games are to be played back to back in less than one week, Greg cannot count on any pitcher to start in more than one game.

Greg knows the strengths and weaknesses not only of his pitchers but also of his opponents, and he believes he can estimate the probability of winning each of the four games with each of the four starting pitchers. Those probabilities are listed in the table at the top of the next column.

Table for Problem 5-22

PLANT	SUPPLY
A	400
B	500
C	350

STORE	DEMAND
X	400
Y	325
Z	400

	TO		
FROM	J	K	L
A	$4	$7	$5
B	$8	$5	$4
C	$5	$6	$7

	TO					
FROM	X	Y	Z	J	K	L
J	$6	$4	$8		$6	$5
K	$3	$6	$7	$6		$7
L	$2	$4	$5	$5	$7	

	OPPONENT			
STARTING PITCHER	DES MOINES	DAVEN-PORT	OMAHA	PEORIA
Jones	0.40	0.80	0.50	0.60
Baker	0.30	0.40	0.80	0.70
Parker	0.80	0.80	0.70	0.90
Wilson	0.20	0.30	0.40	0.50

What pitching rotation should Greg set to provide the highest sum of the probabilities of winning each game for his team?

5-25 Cindy Jefferson, hospital administrator at Anderson Hospital must appoint head nurses to four newly established departments: urology, cardiology, orthopedics, and pediatrics. Believing in the decision modeling approach to problem solving, Cindy has interviewed four nurses—Morris, Richards, Cook, and Morgan—and developed an index scale ranging from 0 to 100 to be used in the assignment. An index of 0 implies that the nurse would be perfectly suited to that task. A value close to 100, on the other hand, implies that the nurse is not at all suited to head that unit. The following table gives the complete set of index scales that Cindy feels represent all possible assignments.

	DEPARTMENT			
NURSE	UROLOGY	CARDIO-LOGY	ORTHO-PEDICS	PEDIATRICS
Morris	15	18	28	75
Richards	23	48	32	38
Cook	24	36	51	36
Morgan	55	38	25	12

Which nurse should be assigned to which unit?

5-26 A trauma center keeps ambulances at locations throughout the east side of a city in an attempt to minimize the response time in the event of an emergency. The times, in minutes, from the ambulance locations to the population centers are given in the table at the top of the next column.

	POPULATION CENTERS			
AMBULANCE LOCATIONS	EAST	NORTH-EAST	SOUTH-EAST	CENTRAL
Site 1	12	8	9	13
Site 2	10	9	11	10
Site 3	11	12	14	11
Site 4	13	11	12	9

Find the optimal assignment of ambulances to population centers that will minimize the total emergency response time.

5-27 The Central Police Department has five detective squads available for assignment to five open crime cases. The chief of detectives wishes to assign the squads so that the total time to conclude the cases is minimized. The average number of days, based on past performance, for each squad to complete each case is shown in following table.

	CASE				
SQUAD	A	B	C	D	E
1	27	7	3	7	14
2	30	6	12	7	20
3	21	5	4	3	10
4	21	12	7	12	8
5	8	26	24	25	13

Use the assignment model to find the best solution.

5-28 Kelly Spaugh, course scheduler of a technical college's business department, needs to assign instructors to courses next semester. As a criterion for judging who should teach each course, Kelly reviews the student evaluations of teaching for the past two years. Because each of the four professors taught each of the four courses at one time or another during the two-year period, Kelly is able to determine a course rating for each instructor. These ratings are shown in the table at the bottom of this page.

Find the best assignment of professors to courses to maximize the overall teaching ratings.

Table for Problem 5-28

	COURSE			
PROFESSOR	STATISTICS	MANAGEMENT	FINANCE	ECONOMICS
Strausbaugh	70	60	80	75
Kelley	80	60	80	75
Davidson	65	55	80	60
Merkle	95	40	65	55

5-29 Coogan Construction is in the process of installing power lines to a large housing development. Rob Coogan wants to minimize the total length of wire used, which will minimize his costs. The housing development is shown as a network in Figure 5.19. Each house has been numbered, and the distance between houses is given in hundreds of feet. What do you recommend?

5-30 The city of Six Mile, South Carolina, is considering making several of its streets one way. What is the maximum number of cars per hour that can travel from east (node 1) to west (node 8)? The network is shown in Figure 5.20.

5-31 Two Chaps and a Truck Movers have been hired to move the office furniture and equipment of Wray Properties to the company's new headquarters. What route do you recommend? The network of roads is shown in Figure 5.21.

5-32 A security firm needs to connect alarm systems to the firm's main control site from five potential trouble locations. Since the systems must be fail-safe, the cables must be run in special pipes. These pipes are very expensive but large enough to simultaneously handle five cables (the maximum that might be needed). Use the minimal-spanning tree model to find the minimum total length of pipes needed to

FIGURE 5.19
Network for Problem 5-29: Coogan Construction

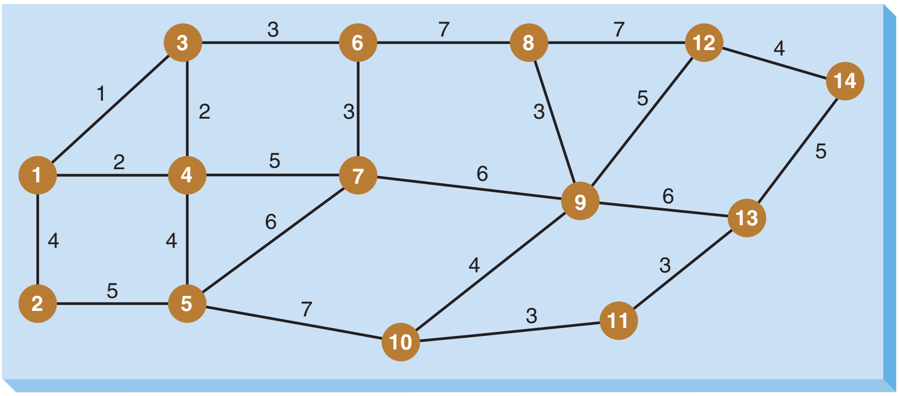


FIGURE 5.20
Network for Problem 5-30: Six Mile, South Carolina

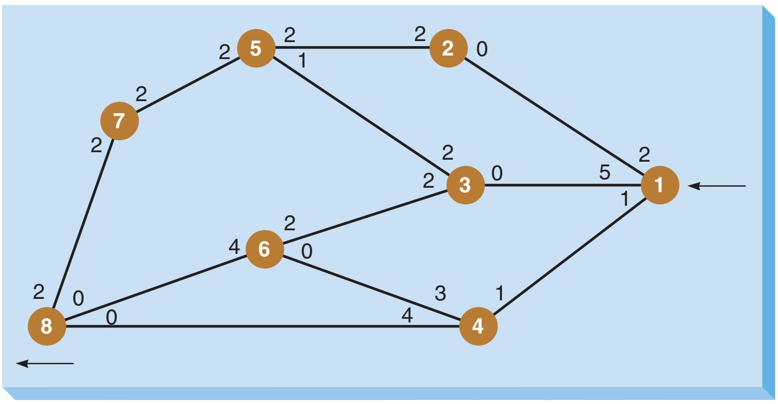
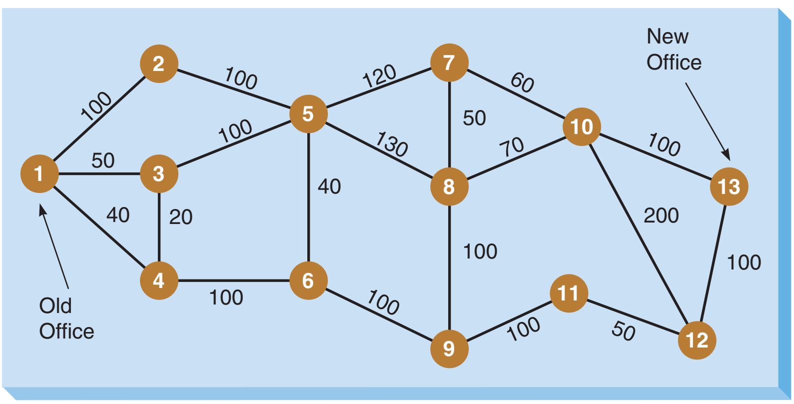


FIGURE 5.21
Network for Problem 5-31: Two Chaps and a Truck Movers



connect the locations shown in Figure 5.22. Node 6 represents the main control site.

5-33 Figure 5.23 shows a network of nodes. Any sequence of activities that takes a flow of one unit from node 1 to node 6 will produce a widget. For example, one unit flowing from node 1 to node 4 to node 6 would create a widget. Other paths are possible. Quantities given are numbers of widgets per day.
(a) How many widgets could be produced in one day?
(b) Suppose we want to ensure that no more than 100 widgets are processed along any of the arcs in the production facility. How many widgets are now possible?

5-34 The road system around the hotel complex (node 1) near a large amusement park (node 11) is shown in Figure 5.24. The numbers by the nodes represent the traffic flow in hundreds of cars per hour. What is the maximum flow of cars from the hotel complex to the park?

5-35 The network in Figure 5.25 shows the pipeline transportation system for treated water from the treatment plant (node 1) to a city water supply system (node 14). The arc capacities represent millions of gallons per hour. How much water can be transported per hour from the plant to the city using this network?

5-36 In Problem 5-35, two of the terminals in the water supply network (see Figure 5.25), represented by nodes 10 and 11, are to be taken offline for routine maintenance. No material can flow in to or out of these nodes. What impact does this have on the capacity of the network? By how much, if any, will the capacity of this network be decreased during the maintenance period?

5-37 The network shown in Figure 5.26 represents the major roads between Port Huron (node 1) and Dearborn (node 14). The values on the arcs represent the distance, in miles. Find the shortest route between the two cities.

5-38 In Problem 5-37, all roads leading into and out of nodes 4 and 9 (see Figure 5.26) have been closed because of bridge repairs. What impact (if any) will this have on the shortest route between Port Huron and Dearborn?

FIGURE 5.22 Network for Problem 5-32

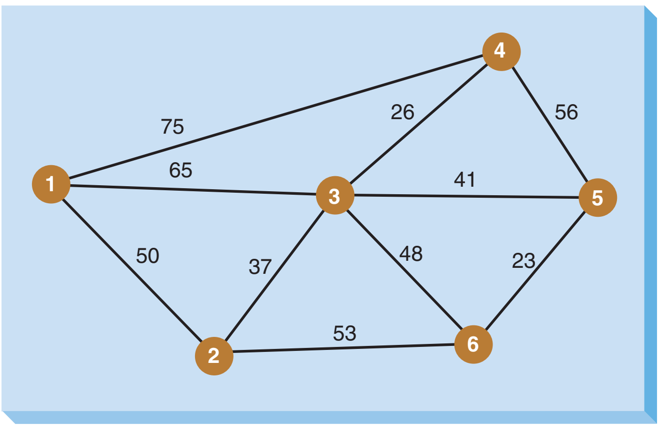


FIGURE 5.23 Network for Problem 5-33

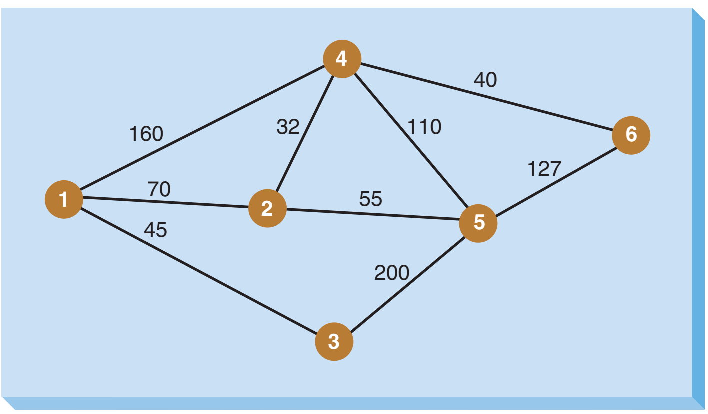


FIGURE 5.24 Network for Problem 5-34

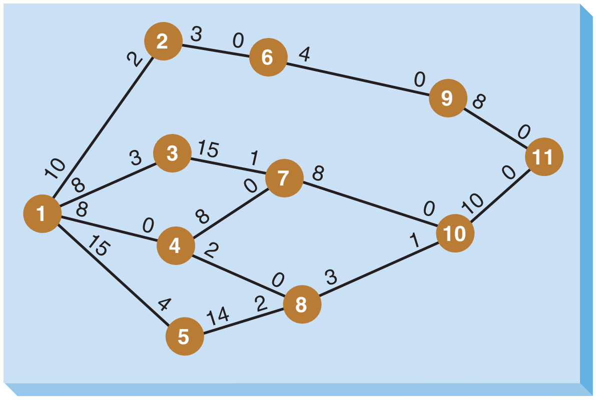


FIGURE 5.25 Network for Problem 5-35

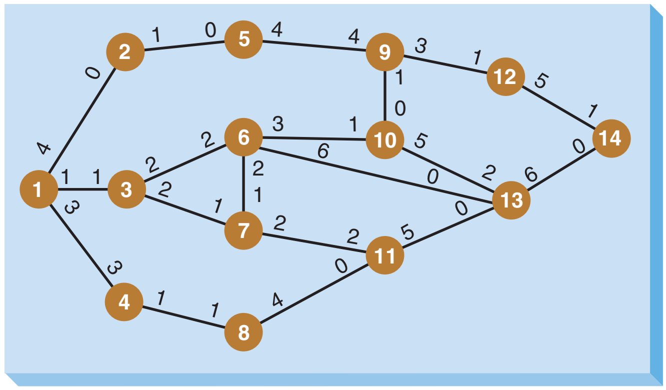


FIGURE 5.26 Network for Problem 5-37

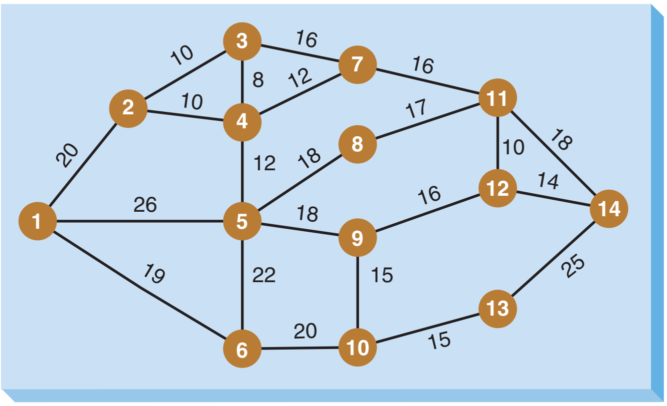
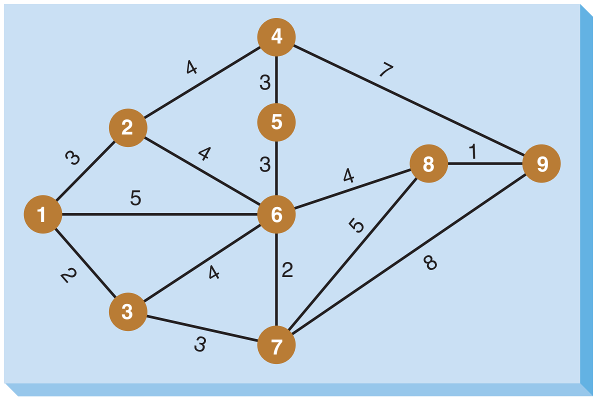


FIGURE 5.27 Network for Problem 5-39



5-39 Solve the minimal-spanning tree model in the network shown in Figure 5.27. Assume that the numbers in the network represent distance in hundreds of yards.

5-40 A secure facility needs to run a hard-wired local area network to connect each of nine sectors. The possible routes the network could utilize, along with the expenses of running the cable between the sectors, in thousands of dollars, are shown in the table at the bottom of this page. Determine the least expensive way to route the network cable to connect all nine sectors. The blank boxes represent no feasible route between the sectors.

5-41 The Kimten Manufacturing Company needs to process 18 jobs. Kimten can process each job on any of the six machines it has available. Iqbal Ahmed, production supervisor at Kimten, wants to allocate jobs to machines in such a manner that all machines have total loads that are as close as possible to each other. For purposes of solving this problem, assume that a single job can be split between multiple machines and that jobs can be moved from one machine to another with no loss of time. The machine hours required for each job are shown in the table in the next column.

JOB	MACHINE HOURS
1	39.43
2	27.94
3	40.27
4	18.27
5	12.72
6	35.20
7	27.94
8	17.20
9	17.20
10	17.20
11	21.50
12	16.74
13	28.29
14	6.07
15	16.01
16	18.49
17	12.16
18	36.67

Table for Problem 5-40

FROM/TO	SECTOR 2	SECTOR 3	SECTOR 4	SECTOR 5	SECTOR 6	SECTOR 7	SECTOR 8	SECTOR 9
Sector 1	6			3	6			
Sector 2		7	6		4			4
Sector 3				4	5	3		8
Sector 4					8		9	3
Sector 5							7	
Sector 6						8	10	
Sector 7								9
Sector 8								2

Case Study

Old Oregon Wood Store

In 2005, George Brown started the Old Oregon Wood Store to manufacture Old Oregon tables. Each table is carefully constructed by hand, using the highest-quality oak. An Old Oregon table can support more than 500 pounds, and since the start of the Old Oregon Wood Store, not one table has been returned because of faulty workmanship or structural problems. In addition to being rugged, each table is beautifully finished, using a urethane varnish that George developed during 20 years of working with wood-finishing materials.

The manufacturing process consists of four steps: preparation, assembly, finishing, and packaging. Each step is performed by one person. In addition to overseeing the entire operation, George does all the finishing. Tom Surowski performs the preparation step, which involves cutting and forming the basic components of the tables. Leon Davis is in charge of the assembly, and Cathy Stark performs the packaging.

Although each person is responsible for only one step in the manufacturing process, everyone can perform any one of the steps. It is George's policy that occasionally everyone should complete several tables on his or her own, without any help or assistance. A small competition is used to see who can complete an entire table in the least amount of time. George maintains average total and intermediate completion times. The data are shown in Figure 5.28.

It takes Cathy longer than the other employees to construct an Old Oregon table. In addition to being slower than the other employees, Cathy is also unhappy about her current responsibility of packaging, which leaves her idle most of the day. Her first preference is finishing, and her second preference is preparation.

In addition to quality, George is concerned about costs and efficiency. When one of the employees misses a day, it causes major scheduling problems. In some cases, George assigns another employee overtime to complete the necessary work. At other times, George simply waits until the employee returns to work to complete his or her step in the manufacturing process. Both solutions cause problems. Overtime is expensive, and waiting causes delays and sometimes stops the entire manufacturing process.

To overcome some of these problems, Randy Lane was hired. Randy's major duties are to perform miscellaneous jobs and to help out if one of the employees is absent. George has given Randy training in all phases of the manufacturing process, and he is pleased with the speed at which Randy has been able to learn how to completely assemble Old Oregon tables. Total and intermediate completion times are given in Figure 5.29.

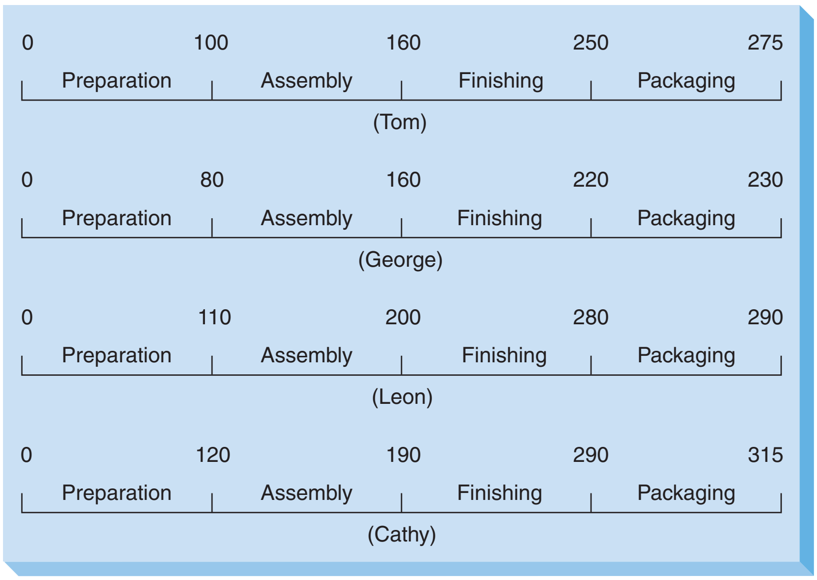


FIGURE 5.28 **Manufacturing Time, in Minutes**

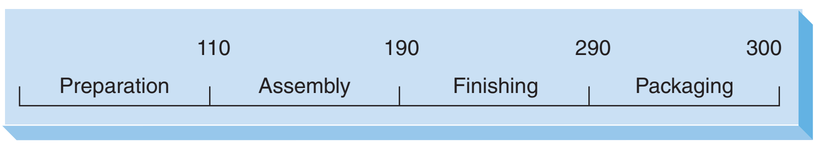


FIGURE 5.29 **Randy's Completion Times, in Minutes**

Discussion Questions

1. What is the fastest way to manufacture Old Oregon tables using the original crew? How many tables could be made per day?
2. Would production rates and quantities change significantly if George would allow Randy to perform one of the four functions and make one of the original crew members the backup person?
3. What is the fastest time to manufacture a table with the original crew if Cathy is moved to either preparation or finishing?
4. Whoever performs the packaging function is severely underutilized. Can you find a better way of utilizing the four- or five-person crew than either giving each a single job or allowing each to manufacture an entire table? How many tables could be manufactured per day with this new scheme?

Case Study

Custom Vans Inc.

Custom Vans Inc. specializes in converting standard vans into campers. Depending on the amount of work and customizing to be done, the customizing could cost less than $1,000 to more than $5,000. In less than four years, Tony Rizzo was able to expand his small operation in Gary, Indiana, to other major outlets in Chicago, Milwaukee, Minneapolis, and Detroit.

Innovation was the major factor in Tony's success in converting a small van shop into one of the largest and most profitable custom van operations in the Midwest. Tony seemed to have a special ability to design and develop unique features and devices that were always in high demand by van owners. An example was Shower-Rific, which Tony developed only six months after he started Custom Vans Inc. These small showers were completely self-contained, and they could be placed in almost any type of van and in a number of different locations within a van. Shower-Rific was made of fiberglass and contained towel racks, built-in soap and shampoo holders, and a unique plastic door. Each Shower-Rific took 2 gallons of fiberglass and 3 hours of labor to manufacture.

Most of the Shower-Rifics were manufactured in Gary, in the same warehouse where Custom Vans Inc. was founded. The manufacturing plant in Gary could produce 300 Shower-Rifics in a month, but that capacity never seemed to be enough. Custom Vans shops in all locations were complaining about not getting enough Shower-Rifics, and because Minneapolis was farther away from Gary than the other locations, Tony was always inclined to ship Shower-Rifics to the other locations before Minneapolis. This infuriated the manager of Custom Vans at Minneapolis, and after many heated discussions, Tony decided to start another manufacturing plant for Shower-Rifics at Fort Wayne, Indiana. The manufacturing plant at Fort Wayne could produce 150 Shower-Rifics per month.

The manufacturing plant at Fort Wayne was still not able to meet current demand for Shower-Rifics, and Tony knew that the demand for his unique camper shower would grow rapidly in the next year. After consulting with his lawyer and banker, Tony concluded that he should open two new manufacturing plants as soon as possible. Each plant would have the same capacity as the Fort Wayne manufacturing plant. An initial investigation into possible manufacturing locations was made, and Tony decided that the two new plants should be located in Detroit, Michigan; Rockford, Illinois; or Madison, Wisconsin. Tony knew that selecting the best location for the two new manufacturing plants would be difficult. Transportation costs and demands for the various locations were important considerations.

The Chicago shop was managed by Bill Burch. This Custom Vans shop was one of the first established by Tony, and it continued to outperform the other locations. The manufacturing plant at Gary was supplying the Chicago shop with 200 Shower-Rifics each month, although Bill knew that the demand for the showers in Chicago was 300 units. The transportation cost per unit from Gary was $10, and although the transportation cost from Fort Wayne was double that amount, Bill was always pleading with Tony to get an additional 50 units from the Fort Wayne manufacturer. The two additional manufacturing plants would certainly be able to supply Bill with the additional 100 showers he needed. The transportation costs would, of course, vary, depending on which two locations Tony picked. The transportation cost per shower would be $30 from Detroit, $5 from Rockford, and $10 from Madison.

Wilma Jackson, manager of the Custom Vans shop in Milwaukee, was the most upset about not getting an adequate supply of showers. She had a demand for 100 units, and at the present time, she was getting only half of that demand from the Fort Wayne manufacturing plant. She could not understand why Tony didn't ship her all 100 units from Gary. The transportation cost per unit from Gary was only $20, while the transportation cost from Fort Wayne was $30. Wilma was hoping that Tony would select Madison as one of the manufacturing locations. She would be able to get all the showers needed, and the transportation cost per unit would be only $5. If not Madison, a new plant in Rockford would be able to supply her total needs, but the transportation cost per unit would be twice as much as it would be from Madison. Because the transportation cost per unit from Detroit would be $40, Wilma speculated that even if Detroit became one of the new plants, she would not be getting any units from Detroit.

Custom Vans Inc. of Minneapolis was managed by Tom Poanski. He was getting 100 showers from the Gary plant. Demand was 150 units. Tom faced the highest transportation costs of all locations. The transportation cost from Gary was

$40 per unit. It would cost $10 more if showers were sent from the Fort Wayne location. Tom was hoping that Detroit would not be one of the new plants, as the transportation cost would be $60 per unit. Rockford and Madison would have costs of $30 and $25, respectively, to ship one shower to Minneapolis.

The Detroit shop's position was similar to Milwaukee's—getting only half of the demand each month. The 100 units that Detroit did receive came directly from the Fort Wayne plant. The transportation cost was only $15 per unit from Fort Wayne, whereas it was $25 from Gary. Dick Lopez, manager of Custom Vans Inc. of Detroit, placed the probability of having one of the new plants in Detroit fairly high. The factory would be located across town, and the transportation cost would be only $5 per unit. He could get 150 showers from the new plant in Detroit and the other 50 showers from Fort Wayne. Even if Detroit was not selected, the other two locations were not intolerable. Rockford had a transportation cost per unit of $35, and Madison had a transportation cost of $40.

Tony pondered the dilemma of locating the two new plants for several weeks before deciding to call a meeting of all the managers of the van shops. The decision was complicated, but the objective was clear—to minimize total costs. The meeting was held in Gary, and everyone was present except Wilma.

Tony: Thank you for coming. As you know, I have decided to open up two new plants at Rockford, Madison, or Detroit. The two locations, of course, will change our shipping practices, and I sincerely hope that they will supply you with the Shower-Rifics that you have been wanting. I know you could have sold more units, and I want you to know that I am sorry for this situation.

Dick: Tony, I have given this situation a lot of consideration, and I feel strongly that at least one of the new plants should be located in Detroit. As you know, I am now getting only half of the showers that I need. My brother, Leon, is very interested in running the plant, and I know he would do a good job.

Tom: Dick, I am sure that Leon could do a good job, and I know how difficult it has been since the recent layoffs by the auto industry. Nevertheless, we should be considering total costs and not personalities. I believe that the new plants should be located in Madison and Rockford. I am farther away from the other plants than any other shop, and these locations would significantly reduce transportation costs.

Dick: That may be true, but there are other factors. Detroit has one of the largest suppliers of fiberglass, and I have checked prices. A new plant in Detroit would be able to purchase fiberglass for $2 per gallon less than any of the other existing or proposed plants.

Tom: At Madison, we have an excellent labor force. This is due primarily to the large number of students attending the University of Madison. These students are hard workers, and they will work for $1 less per hour than the other locations that we are considering.

Bill: Calm down, you two. It is obvious that we will not be able to satisfy everyone in locating the new plants. Therefore, I would like to suggest that we vote on the two best locations.

Tony: I don't think that voting would be a good idea. Wilma was not able to attend, and we should be looking at all these factors together in some type of logical fashion.

Discussion Question

Where would you locate the two new plants?

Case Study

Binder's Beverage

Bill Binder's business nearly went under when Colorado almost passed the bottle bill. Binder's Beverage produced soft drinks for many of the large grocery stores in the area. After the bottle bill failed, Binder's Beverage flourished. In a few short years, the company had a major plant in Denver and a warehouse in east Denver. The problem was getting the finished product to the warehouse. Although Bill was not good with distances, he was good with times. Denver is a big city with numerous roads that can be taken from the plant to the warehouse. Figure 5.30 shows the road network.

The soft drink plant is located at the corner of North Street and Columbine Street. High Street also intersects North Street and Columbine Street at the plant. Twenty minutes due north of the plant on North Street is I-70, the major east–west highway in Denver.

North Street intersects I-70 at Exit 135. It takes 5 minutes driving east on I-70 to reach Exit 136. This exit connects I-70 with High Street and 6th Avenue. Ten minutes east on I-70 is Exit 137. This exit connects I-70 with Rose Street and South Avenue.

From the plant, it takes 20 minutes on High Street, which goes in a northeast direction, to reach West Street. It takes another 20 minutes on High Street to reach I-70 and Exit 136.

It takes 30 minutes on Columbine Street to reach West Street from the plant. Columbine Street travels east and slightly north.

West Street travels east and west. From High Street, it takes 15 minutes to get to 6th Avenue on West Street. Columbine Street also comes into this intersection. From this intersection, it takes an additional 20 minutes on West Street to get to Rose Street, and it takes another 15 minutes to get to South Avenue.

From Exit 136 on 6th Avenue, it takes 5 minutes to get to West Street. Sixth Avenue continues to Rose Street, requiring 25 minutes. Sixth Avenue then goes directly to the warehouse.

FIGURE 5.30
Road Map for Binder's Beverage

From Rose Street, it takes 40 minutes to get to the warehouse on 6th Avenue.

At Exit 137, Rose Street travels southwest. It takes 20 minutes to intersect with West Street, and it takes another 20 minutes to get to 6th Avenue. From Exit 137, South Street goes due south. It takes 10 minutes to get to West Street and another 15 minutes to get to the warehouse.

Discussion Question

What route do you recommend?

Internet Case Studies

See the Companion Website for this textbook, at www.pearsonhighered.com/balakrishnan, for additional case studies.

CHAPTER 6

Integer, Goal, and Nonlinear Programming Models

LEARNING OBJECTIVES

After completing this chapter, students will be able to:

1. Formulate integer programming (IP) models.
2. Set up and solve IP models using Excel's Solver.
3. Understand the difference between general integer and binary integer variables.
4. Understand the use of binary integer variables in formulating problems involving fixed costs.
5. Formulate goal programming (GP) problems and solve them using Excel's Solver.
6. Formulate nonlinear programming (NLP) problems and solve them using Excel's Solver.

CHAPTER OUTLINE

6.1 Introduction

Earlier chapters focus on the linear programming (LP) category of mathematical programming models. These LP models have three characteristics:

- The decision variables are allowed to have fractional values.
- There is a unique objective function.
- All mathematical expressions (objective function and constraints) have to be linear.

This chapter presents a series of other important mathematical models that allow us to relax each of these basic LP conditions. The new models—integer programming, goal programming, and nonlinear programming—are introduced here and then discussed in detail in the remainder of this chapter.

Integer Programming Models

Integer programming *is an extension of LP that solves problems that require integer solutions.*

General integer variables *can take on any nonnegative integer value.*

Binary variables *must equal either 0 or 1.*

Although fractional values such as $X = 0.33$ and $Y = 109.4$ may be valid for decision variables in many problems, a large number of business problems can be solved only if variables have *integer* values. For example, when an airline decides how many flights to operate on a given sector, it can't decide to operate 5.38 flights; it must operate 5, 6, or some other integer number.

In sections 6.2 and 6.3, we present two types of integer variables: general integer variables and binary variables. **General integer variables** are variables that can take on any nonnegative integer value that satisfies all the constraints in a model (e.g., 5 submarines, 8 employees, 20 insurance policies). **Binary variables** are a special type of integer variables that can take on only either of two values: 0 or 1. In this chapter we examine how problems involving both of these types of integer variables can be formulated and solved using Excel's Solver.

Integer programming (IP) problems can also be classified as *pure* and *mixed* types of problems, as follows:

- ***Pure IP problems.*** These are problems in which all decision variables must have integer solutions (general integer, binary, or a combination of the two).
- ***Mixed IP problems.*** These are problems in which some, but not all, decision variables must have integer solutions (i.e., general integer, binary, or a combination of the two). The noninteger variables can have fractional optimal values. We discuss an example of these types of problems in section 6.4.

Goal Programming Models

Goal programming *is an extension of LP that permits more than one objective to be stated.*

LP forces a decision maker to state only one objective. But what if a business has several objectives? Management may indeed want to minimize costs, but it might also simultaneously want to maximize market share, maximize machine utilization, maintain full employment, and minimize environmental impacts. These objectives can often conflict with each other. For example, minimizing costs may be in direct conflict with maintaining full employment. Goal programming is an extension to LP that permits multiple objectives such as these to be considered simultaneously. We discuss goal programming in detail in section 6.5.

Nonlinear Programming Models

With **nonlinear programming**, *objectives and/or constraints are nonlinear.*

Linear programming can, of course, be applied only to cases in which the objective function and all constraints are linear expressions. Yet in many situations, this may not be the case. For example, consider a price curve that relates the unit price of a product to the number of units made. As more units are made, the price per unit may decrease in a nonlinear fashion. Hence, if X and Y denote the number of units of two products to make, the objective function could be

$$\text{Maximize profit} = 25X - 0.4X^2 + 30Y - 0.5Y^2$$

Because of the squared terms, this is a nonlinear objective function. In a similar manner, we could have one or more nonlinear constraints in the model. We discuss nonlinear programming models in detail in section 6.6.

Now let's examine each of these extensions of LP—integer, goal, and nonlinear programming—one at a time.

6.2 Models with General Integer Variables

Models with general integer variables are similar to LP models—except that variables must be integer valued.

A model with general integer variables (which we will call an *IP model*) has an objective function and constraints identical to those of LP models. There is no real difference in the basic procedures for formulating an IP model and an LP model. The only additional requirement in an IP model is that one or more of the decision variables must take on integer values in the optimal solution. The actual value of this integer variable is, however, limited only by the constraints in the model. That is, values such as 0, 1, 2, 3, and so on are perfectly valid for these variables, as long as these values satisfy all constraints in the model.

Let us look at a simple two-variable example of an IP problem and see how to formulate it. We recognize that you are unlikely to ever encounter such small problems in real-world situations. However, as discussed in Chapter 2, a primary advantage of two-variable models is that we can easily represent them on a two-dimensional graph and use them effectively to illustrate how IP models behave. Let us then also look at how this two-variable IP model can be set up and solved by using Excel's Solver. The Excel setup can be extended to handle much larger IP models. In fact, thanks to the continued significant advances in computing technology, researchers have successfully modeled and solved IP models involving thousands of decision variables and constraints in just a few minutes (or even seconds).

Harrison Electric Company

Example of an IP model: Harrison Electric

Harrison Electric Company, located in Chicago's Old Town area, produces two expensive products that are popular with renovators of historic old homes: ornate lamps and old-fashioned ceiling fans. Both lamps and ceiling fans require a two-step production process involving wiring and assembly time. It takes about 2 hours to wire each lamp and 3 hours to wire a ceiling fan. Final assembly of each lamp and fan requires 6 and 5 hours, respectively. The production capability this period is such that only 12 hours of wiring time and 30 hours of assembly time are available. Each lamp produced nets the firm \$600 and each fan nets \$700 in profit.

FORMULATING THE PROBLEM If we let L denote the number of lamps to make and F denote the number of ceiling fans to make, Harrison's product mix decision can be formulated using LP as follows:

$$\text{Maximize profit} = \$600L + \$700F$$

subject to the constraints

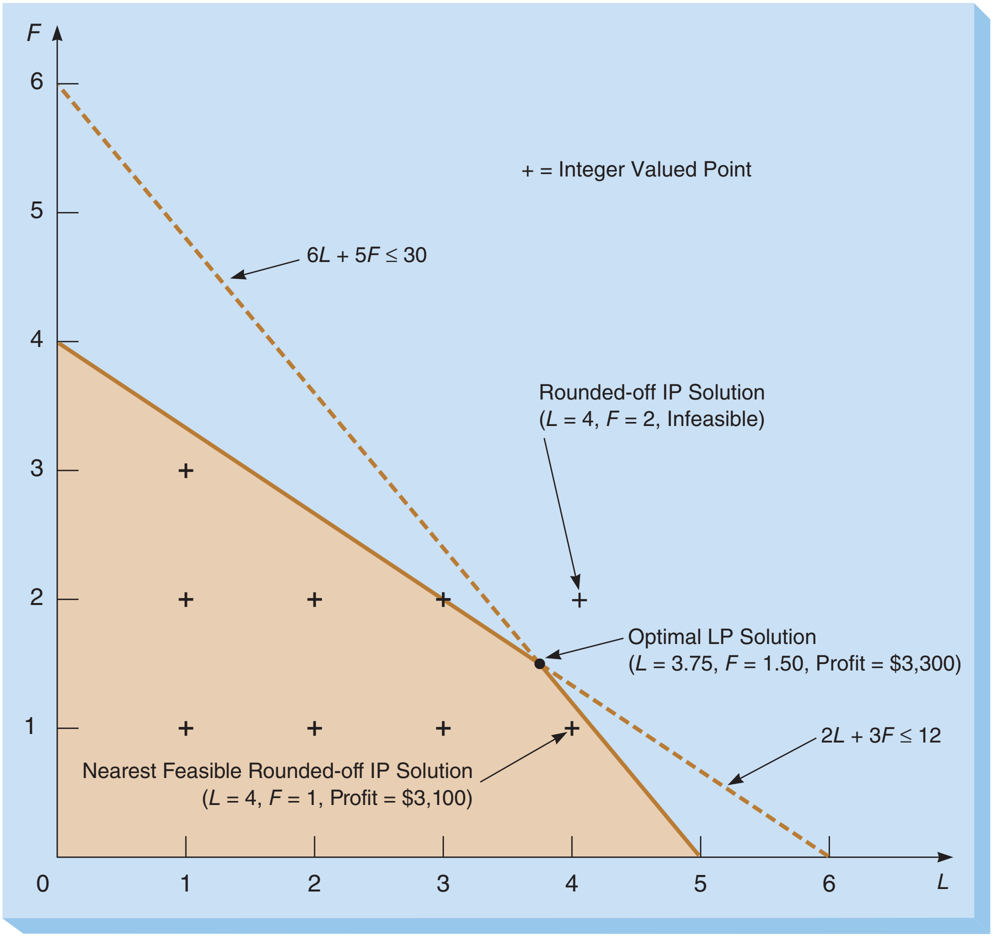
$$\begin{aligned} 2L + 3F &\leq 12 && \text{(wiring hours)} \\ 6L + 5F &\leq 30 && \text{(assembly hours)} \\ L, F &\geq 0 \end{aligned}$$

SOLVING THE PROBLEM GRAPHICALLY Because there are only two decision variables, let us employ the graphical approach to visualize the feasible region. The shaded region in Figure 6.1 shows the feasible region for the LP problem. The optimal corner point solution turns out to be $L = 3.75$ lamps and $F = 1.50$ ceiling fans, for a profit of \$3,300.

Rounding is one way to reach integer solution values, but it often does not yield the optimal IP solution.

INTERPRETING THE RESULTS Because Harrison cannot produce and sell a fraction of a product, the production planner, Wes Wallace, recognized that he was dealing with an IP problem. It seemed to Wes that the simplest approach was to round off the optimal fractional LP solutions for L and F to integer values. Unfortunately, rounding can produce two problems. First, if we use the traditional rounding rule (i.e., round down if the fraction is below 0.5 and round up otherwise), the resulting integer solution may not even be in the feasible region. For example, using this rule, we would round Harrison's LP solution to $L = 4$ lamps and $F = 2$ fans. As we can see from Figure 6.1, that solution is not feasible. It is, hence, not a practical answer. Second, even if we manage to round the LP solution in such a way that the resulting integer solution is feasible, there is no guarantee that it is the optimal IP solution. For example, suppose Wes considers all possible integer solutions to Harrison's problem (shown in Figure 6.1) and rounds the LP solution to its nearest feasible IP solution (i.e., $L = 4$ lamps and $F = 1$ fan). As we will see later, it turns out that this IP

FIGURE 6.1
Graph for Harrison Electric—General IP



solution is not the optimal solution. Also, note that because this problem involves only two variables, we are able to at least visualize which IP solutions are feasible and round off the LP solution appropriately. Obviously, even the process of rounding the LP solution to obtain a feasible IP solution could be very cumbersome to do if there are more variables in the model.

What is the optimal integer solution in Harrison's case? Table 6.1 lists the entire set of integer-valued solutions for this problem. By inspecting the right-hand column, we see that the optimal integer solution is $L = 3$ lamps and $F = 2$ ceiling fans, for a total profit $=$ \$3,200. The IP solution of $L = 4$ lamps and $F = 1$ fan yields a profit of only \$3,100.

An important concept to understand is that an IP solution can never be better than the solution to the same LP problem. The integer problem is usually worse in terms of higher cost or lower profit.

PROPERTIES OF OPTIMAL INTEGER SOLUTIONS We note two important properties of the optimal integer solution. First, the optimal point $L = 3$ and $F = 2$ is not a corner point (i.e., a point where two or more constraints intersect) in the LP feasible region. In fact, unlike LP problems, in which the optimal solution is always a corner point of the feasible region, the optimal solution in an IP model need not be a corner point. As we will discuss shortly, this is what makes it difficult to solve IP models in practice.

Second, the integer restriction results in an objective function value that is no better (and is usually worse) than the optimal LP solution. The logic behind this occurrence is quite simple. The feasible region for the original LP problem includes *all* IP solution points, in addition to several LP solution points. That is, the optimal IP solution will always be a feasible solution for the LP problem, *but not vice versa*. We call the LP equivalent of an IP problem (i.e., the IP model with the integer requirement deleted) the *relaxed* problem. As a rule, the IP solution can never produce a better objective value than its LP relaxed problem. At best, the two solutions can be equal (if the optimal LP solution turns out to be integer valued).

TABLE 6.1
Integer Solutions for Harrison Electric—General IP

LAMPS (L)	CEILING FANS (F)	PROFIT ($\$600L + \$700F$)	
0	0	$ 0	
1	0	$ 600	
2	0	$1,200	
3	0	$1,800	
4	0	$2,400	
5	0	$3,000	
0	1	$ 700	
1	1	$1,300	
2	1	$1,900	
3	1	$2,500	
4	1	$3,100	← *Nearest feasible rounded-off IP solution*
0	2	$1,400	
1	2	$2,000	
2	2	$2,600	
3	2	$3,200	← *Optimal IP solution*
0	3	$2,100	
1	3	$2,700	
0	4	$2,800	

Although using enumeration is feasible for some small IP problems, it can be difficult or impossible for large ones.

Although it is possible to solve simple IP problems such as Harrison Electric's by inspection or enumeration, larger problems cannot be solved in this manner. There would simply be too many points to enumerate. Fortunately, most LP software packages, including Excel's Solver, are capable of handling models with integer variables.

The Excel layout for IP models is similar to that used for LP models.

Using Solver to Solve Models with General Integer Variables

We can set up Harrison Electric's IP problem in Excel in exactly the same manner as we have done for several LP examples in Chapters 2 and 3. For clarity, we once again use the same Excel

IN ACTION Improving Disaster Response Times at CARE International Using Integer Programming

Each year natural disasters kill about 70,000 people and affect another 200 million people worldwide. When a disaster strikes, large quantities of supplies are needed to provide relief aid to the affected areas. However, unavailability of supplies or slowness in mobilizing them may cause emergency responses to be ineffective, resulting in increased human suffering and loss of life.

CARE International, with programs in 65 countries, is one of the largest humanitarian organizations that provide relief aid to disaster survivors. To improve disaster response times, CARE collaborated with researchers from Georgia Institute of Technology to develop a model that evaluates the effect of pre-positioning relief items on average response times.

The model focuses on up-front investment (initial inventory stocking and warehouse setup) and average response time and seeks to answer the following question: Given an initial investment, which network configuration minimizes the average response time? To answer this question, the researchers developed a mixed-integer programming model. The model included about 470,000 variables, 12 of which are binary, about 56,000 constraints, and yielded optimal solutions in under four hours.

The model's results helped CARE determine a desired configuration for its pre-positioning network. Based in part on the results of the study, CARE has pre-positioned relief supplies in three facilities around the world—Dubai, Panama, and Cambodia.

Source: Based on S. Duran, M. A. Gutierrez, and P. Keskinocak. "Pre-Positioning of Emergency Items for CARE International," *Interfaces* 41, 3 (May–June 2011): 223–237.

layout here as in those chapters; that is, all parameters (solution value, objective coefficients, and constraint coefficients) associated with a decision variable are modeled in the same column. The objective function and each constraint in the model are shown on separate rows of the worksheet.

Excel Notes

- The Companion Website for this textbook, at www.pearsonhighered.com/balakrishnan, contains the Excel file for each sample problem discussed here. The relevant file name is shown in the margin next to each example.
- In each of the Excel layouts, for clarity, changing variable cells are shaded yellow, the objective cell is shaded green, and cells containing the left-hand-side (LHS) formula for each constraint are shaded blue.
- Also, to make the equivalence of the *written* formulation and the Excel layout clear, the Excel layouts show the decision variable names used in the written formulation of the model. Note that these names have no role in using Solver to solve the model.

File: 6-1.xls

The Excel layout for Harrison Electric's problem is shown in Screenshot 6-1. As usual, we specify the objective cell (objective function), changing variable cells (decision variables), and constraint LHS and right-hand-side (RHS) cell references in the Solver Parameters window.

The integer requirement is specified as an additional constraint in Solver.

SPECIFYING THE INTEGER REQUIREMENT Before we solve the model, we need to specify the integer value requirement for the two decision variables. We specify this in Solver as a constraint, as follows:

- Use the Add option to include a new constraint. In the LHS entry for the new constraint (see Screenshot 6-1), enter the cell reference for a decision variable that must be integer valued. If there are several decision variables in the model that must be integer valued and they are in contiguous cells (i.e., next to each other), the entire cell range may be entered in the LHS entry. For Harrison's problem, the entry in this box would be B5:C5, corresponding to the number of lamps and fans to make, respectively.

The **int** *option is used in Solver to specify general integer variables.*

- Next, click the drop-down box in the Add Constraint window. Note that this box has six choices, of which three (i.e., <=, >=, and =) have been used so far. The remaining three are int (for Integer), bin (for Binary), and dif (for AllDifferent).[1] Click the choice int. The word *integer* is displayed automatically in the box for the RHS entry. This indicates to Solver that all variables specified in the LHS box must be integer valued in the optimal solution.

SOLVING THE IP MODEL We are now ready to solve the IP model. Before we click Solve, we need to verify that the Make Unconstrained Variables Non-Negative box is checked and that Simplex LP is specified in the Select a Solving Method box, as shown in Screenshot 6-1. The result, also shown in Screenshot 6-1, indicates that 3 lamps and 2 fans, for a profit of $3,200, is identified as the optimal solution.

As noted previously, thanks to advances in computing technology, Solver (and other decision modeling software packages) can identify optimal solutions very quickly, even for IP models involving thousands of decision variables and constraints. However, when compared to an LP model, the computational effort required to solve an IP model (of the same size) grows rapidly with problem size. We now briefly discuss the reason for this phenomenon.

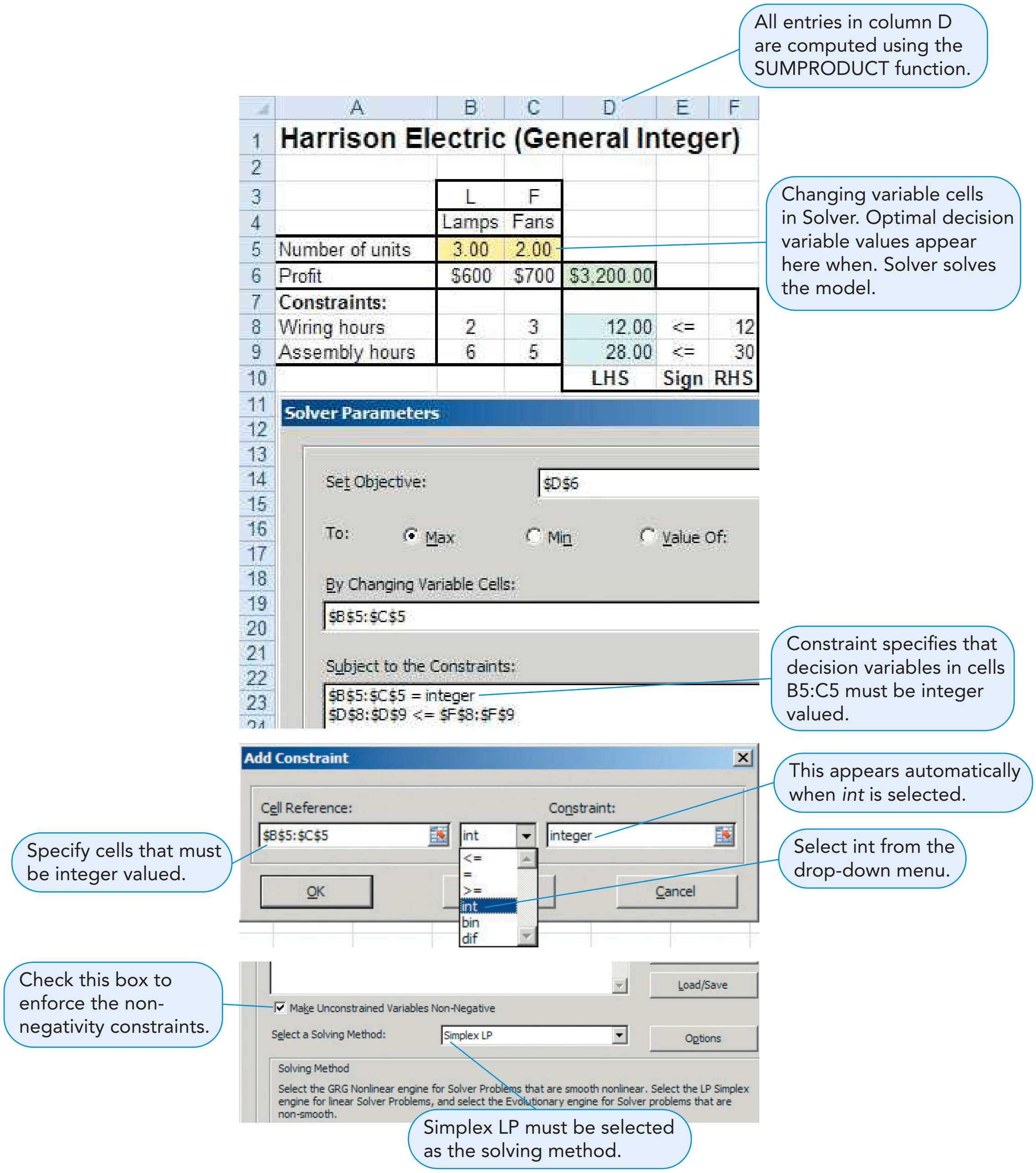
How Are IP Models Solved?

Solver uses the **branch-and-bound** *procedure to solve IP problems.*

As shown in Figure 6.1 on page 214, the optimal solution to an IP model need not be at a corner point of the feasible region. Unfortunately, the simplex method evaluates only corner points as candidates for the optimal solution. In order to use the simplex method to identify an integer-valued optimal point that may *not* be a corner point, we employ a procedure called the **branch-and-bound (B&B) method**. The B&B method is used by most software packages, including Solver, to solve IP models.

[1] We illustrate only the *int* and *bin* choices in this textbook. The third choice, *dif*, is relevant for special types of sequencing models that are not discussed here.

SCREENSHOT 6-1 Excel Layout and Solver Entries for Harrison Electric—General IP



Although we do not discuss the details of the B&B procedure in this textbook, we provide a brief description of how it works.[2] Essentially, the B&B procedure uses a "divide and conquer" strategy. Rather than try to search for the optimal IP solution over the entire feasible region at one time, the B&B procedure splits the feasible region into progressively smaller and smaller subregions. It then searches each subregion in turn. Clearly, the best IP solution over all subregions will be the optimal IP solution over the entire feasible region.

Solving a single IP problem can involve solving several LP problems.

In creating each subregion, the B&B procedure forces a corner point of the new subregion to have integer values for at least one of the variables in the model. This procedure is called *branching*. Finding the optimal solution for each subregion involves the solution of an LP model, referred to in Solver as a subproblem. Hence, in order to solve a single IP model, we may have to solve several LP subproblems. Clearly, this could become computationally quite

[2] Details of the B&B procedure can be found in B. Render, R. Stair, and M. Hanna. *Quantitative Analysis for Management*, 11th ed. Upper Saddle River, NJ: Prentice Hall, 2012.

Computer time and memory requirements may make it difficult to solve large IP models.

burdensome, depending on the number of subregions that need to be created and examined for an IP model. The computer memory needed could also become extensive, especially for models with a large number of integer variables, because we need to store detailed information regarding each subregion (e.g., what part of the LP feasible region does this subregion occupy, has it been examined, is the optimal solution integer). Stopping rules are used to efficiently conduct and stop the search process for different subregions.

Solver Options

Now let us return to how Solver handles IP problems and examine the options available when solving IP models. The Options window is shown in Screenshot 6-2A. We have not concerned ourselves about these options so far while solving LP models because the default values are adequate to solve most, if not all, LP models considered here. However, for IP models, some of these options deserve additional attention.

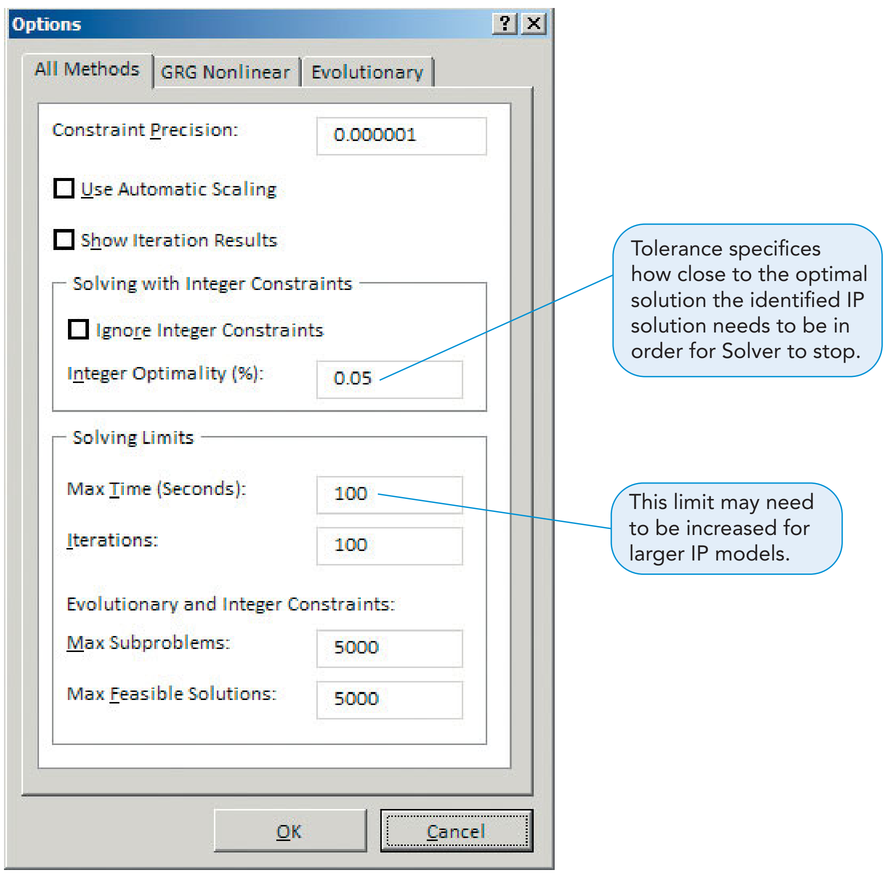
The maximum time allowed could become an issue for large IP problems.

SOLVING WITH INTEGER CONSTRAINTS Checking the Ignore Integer Constraints box causes Solver to solve the problem as an LP model. As discussed earlier, the optimal objective value of an IP model will always be *worse* than that for the corresponding LP model (i.e., lower profit for a maximization problem and higher cost for a minimization problem). Hence, this option allows us to quickly get an idea about the best IP solution that we can find for the model.

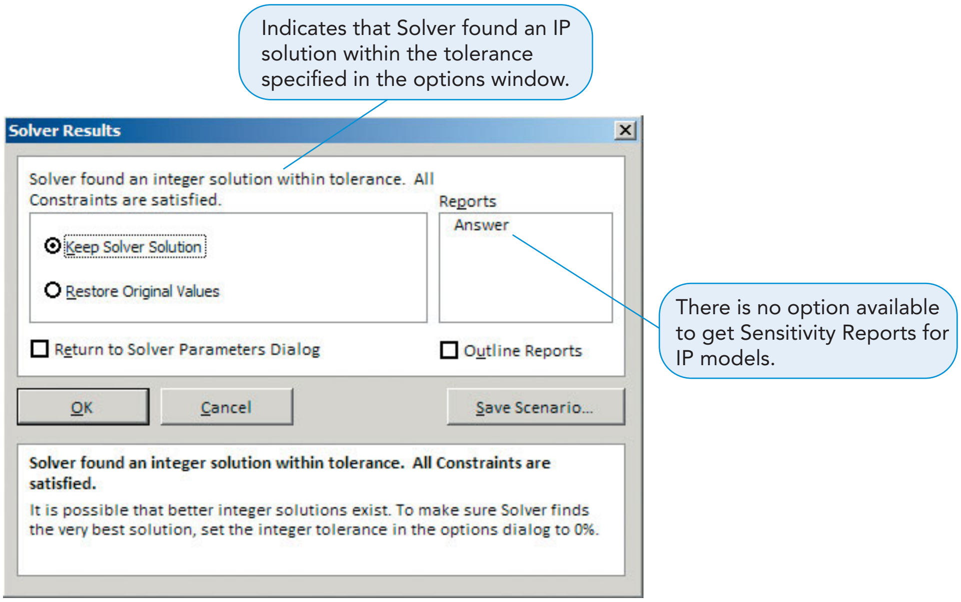
Reducing the tolerance will yield a more accurate IP solution—but could take more time.

The Integer Optimality (%) option is set at a default value of 5% (shown as 0.05 in Solver). A value of 5% implies that we are willing to accept an IP solution that is within 5% of the true optimal IP solution value. When Solver finds a solution within the allowable tolerance, it stops and presents that as the final solution. When this occurs, it is explicitly indicated by the message "Solver found an integer solution within tolerance," as shown in Screenshot 6-2B. If we wish to find the *true* optimal solution, we must set the tolerance to 0%.

SCREENSHOT 6-2A
Solver Options Window for IP Models



SCREENSHOT 6-2B
Solver Results Window for IP Models



SOLVING LIMITS In Screenshot 6-2A, note that the Max Time (Seconds) option is set to a default value of 100 seconds. As the number of integer-valued decision variables increases in an IP model, this time limit may be exceeded and will need to be extended. In practice, however, it is a good idea to keep the limit at its default value and run the problem. Solver will warn you when the limit is reached and give you the opportunity to allow more time for an IP problem to solve.

Likewise, the default values of 5,000 for the Max Subproblems and Max Feasible Solutions options should be sufficient for most, if not all, IP models considered here. Recall from our brief discussion of the B&B method that in order to solve a single IP model, Solver may need to solve several LP subproblems.

Should We Include Integer Requirements in a Model?

We have already discussed one reason we should be cautious about including integer requirements in a model—namely, the possible computational burden involved in solving large IP models. A second reason for this caution has to do with Sensitivity Reports.

Sensitivity Reports are not available for IP models.

Recall from Chapter 4 that after solving an LP model, we can generate a Sensitivity Report that provides a wealth of information regarding how sensitive the current optimal solution is to changes in input data values. The information in this report allows us to even analyze issues such as the impact of acquiring additional resources, pricing out new products, etc. However, as soon as we specify that one or more decision variables in the model are integers, we lose the ability to obtain a Sensitivity Report. In fact, as shown in Screenshot 6-2B, Solver does not even give you an option to get a Sensitivity Report for an IP model.

Do these two reasons mean that we should not include the integer requirements in a model? For many real-world IP models (and all the models discussed in this textbook), the computational issue is probably not relevant due to the available computing technology. However, in practice, it is a good idea to ask ourselves the following question, especially when the model includes a large number of integer decision variables: “Do we definitely need to find the optimal integer solution, or would it be acceptable to solve the problem as an LP problem and then round off, even if that may lead to a slightly suboptimal solution?” Obviously, there is no single easy answer to this question, and the ultimate answer would depend on the cost (or profit) implications for that specific problem. The answer to this question could also be influenced by the desirability of being able to have a Sensitivity Report for the particular problem scenario.

IN ACTION Selling Seats at American Airlines Using Integer Programming

American Airlines (AA) describes *yield management* as "selling the right seats to the right customers at the right prices." The role of yield management is to determine how much of each product to put on the shelf (i.e., make available for sale) at a given point in time.

The AA yield-management problem is a mixed-integer program that requires data such as passenger demand, cancellations, and other estimates of passenger behavior that are subject to frequent changes. To solve the systemwide yield-management problem would require approximately 250 million decision variables.

To bring this problem down to a manageable size, AA's IP model creates three smaller and easier subproblems. The airline looks at

1. Overbooking, which is the practice of intentionally selling more reservations for a flight than there are actual seats on the aircraft
2. Discount allocation, which is the process of determining the number of discount fares to offer on a flight
3. Traffic management, which is the process of controlling reservations by passenger origin and destination to provide the mix of markets that maximizes revenue

Yield management has been a big winner not only for AA and other airlines, but for other service providers such as hotels. Each year, airlines estimate that profits increase by several million dollars due to the use of this approach. Since its introduction over twenty years ago, airlines and large hotel chains have continually worked to refine the yield management process to make it more efficient and profitable for their operations.

Sources: Based on T. Cook. "SABRE Soars," *OR/MS Today* 25, 3 (June 1998): 26–31; and B. Smith, J. Leimkuhler, and R. Darrow. "Yield Management at American Airlines," *Interfaces* 22, 1 (January–February 1992): 8–31.

6.3 Models with Binary Variables

Binary variables are restricted to values of 0 or 1.

As discussed earlier, binary variables are restricted to values of 0 and 1. Recall that the assignment model and shortest-path model in Chapter 5 both involve variables that ultimately take on values of either 0 or 1 at optimality. However, in both of those models, we do not have to explicitly specify that the variables were binary. The integer property of network flow models, along with the supply and demand values of one unit each, automatically ensure that the optimal solution has a value of 0 or 1.

We associate a value of 1 with one of the choices and a value of 0 with the other choice.

In contrast, we now examine models in which we will explicitly specify that the variables are binary. A binary variable is a powerful modeling tool that is applicable whenever we want to model a *yes* or *no* decision between exactly two choices. That is, the decision has to select either choice 1 or choice 2 in its entirety, and partial or fractional selections are not allowed. When we are faced with such a decision, we associate a binary variable with it. With one of the two choices, we associate a value of 1 for the binary variable. A value of 0 for the binary variable is then associated with the other choice. Now, we write the objective function and constraints in a manner that is consistent with this definition of the binary variable.

Two popular applications of binary models are selection and set covering.

A popular application of binary variables is in *selection* problems, which involve the selection of an optimal subset of items from a larger set of items. Typical examples include decisions such as introducing new products (e.g., introduce a specific product or not), building new facilities (e.g., build a specific facility or not), selecting team members (e.g., select a specific individual or not), and investing in projects (e.g., invest in a specific project or not). Another popular application of binary variables is in a class of problems known as *set covering* problems. These problems typically deal with trying to identify the optimal set of locations to cover or serve a specified set of customers. Examples include locating fire stations, police precincts, or medical clinics to serve a community, locating cell phone towers to provide uninterrupted signal over a region, etc.

Let us consider simple examples to illustrate both types of problems—selection and set covering—that use binary variables.

Portfolio Selection at Simkin and Steinberg

Here is an example of stock portfolio selection with 0–1 programming.

The Houston-based investment firm of Simkin and Steinberg specializes in recommending oil stock portfolios for wealthy clients. One such client has up to $3 million available for investments and insists on purchasing large blocks of shares of each company in which he invests. Table 6.2

TABLE 6.2
Oil Investment Opportunities

COMPANY NAME (LOCATION)	EXPECTED ANNUAL RETURN (THOUSANDS)	COST FOR BLOCK OF SHARES (THOUSANDS)
Trans-Texas Oil (Texas)	$ 50	$ 480
British Petro (Foreign)	$ 80	$ 540
Dutch Shell (Foreign)	$ 90	$ 680
Houston Drilling (Texas)	$120	$1,000
Lone Star Petro (Texas)	$110	$ 700
San Diego Oil (California)	$ 40	$ 510
California Petro (California)	$ 75	$ 900

describes the various companies that are under consideration. The objective is to maximize annual return on investment, subject to the following specifications made by the client:

- At least two Texas companies must be in the portfolio.
- No more than one investment can be made in foreign companies.
- Exactly one of the two California companies must be included.
- If British Petro stock is included in the portfolio, then Trans-Texas Oil stock must also be included.

FORMULATING THE PROBLEM Note that the decision with regard to each company has to be one of two choices. That is, the investment firm either buys a large block of shares in the company or it doesn't buy the company's shares. To formulate this problem, let us therefore associate a binary variable with each of the seven companies. For example, we define a binary variable, T, for Trans-Texas Oil as follows:

$$\begin{aligned} T &= 1 \text{ if Trans-Texas Oil is included in the portfolio} \\ &= 0 \text{ if Trans-Texas Oil is not included in the portfolio} \end{aligned}$$

In a similar manner, we define binary variables B (British Petro), D (Dutch Shell), H (Houston Oil), L (Lone Star Petro), S (San Diego Oil), and C (California Petro).

We now need to express the objective function and constraints in a manner that is consistent with the previous definition of the binary variables. The objective function can be written as

$$\begin{aligned} \text{Maximize return on investment} = \$50T + \$80B + \$90D + \$120H \\ + \$110L + \$40S + \$75C \end{aligned}$$

All figures are in thousands of dollars. In the previous expression, if T has an optimal value of 1 (implying that we include Trans-Texas Oil in the portfolio), this would contribute $50,000 to the total return. In contrast, if T has an optimal value of 0 (implying that we *not* include Trans-Texas Oil in the portfolio), this would contribute $0 to the total return.

Next, we model the constraints. The constraint regarding the $3 million investment limit can be expressed in a similar manner to that of the objective function. That is,

$$\$480T + \$540B + \$680D + \$1{,}000H + \$700L + \$510S + \$900C \leq \$3{,}000$$

Again, all figures are in thousands of dollars. Depending on whether the optimal value of a binary variable is 0 or 1, the corresponding investment cost will be calculated in the LHS of the previous expression.

Binary variables can be used to write different types of constraints.

The other constraints in the problems are special ones that exploit the binary nature of these variables. These types of constraints are what make the use of binary variables a powerful modeling tool. We discuss these special constraints in the following sections.

Selecting k out of n choices

***k* OUT OF *n* CHOICES** The requirement that at least two Texas companies must be in the portfolio is an example of a "k out of n choices" constraint. There are three (i.e., $n = 3$) Texas companies (denoted by the variables T, H, and L), of which at least two (i.e., $k = 2$) must be selected. We can model this constraint as

$$T + H + L \geq 2$$

Avoiding incompatible selections

MUTUALLY EXCLUSIVE CHOICES The condition that no more than one investment can be made in foreign companies is an example of a *mutually exclusive* constraint. Note that the inclusion of one foreign company means that the other must be excluded. We can model this constraint as

$$B + D \leq 1$$

The condition regarding the two California companies is also an example of having mutually exclusive variables. The sign of this constraint is, however, an equality rather than an inequality because Simkin and Steinberg *must* include a California company in the portfolio. That is,

$$S + C = 1$$

Enforcing dependencies

IF–THEN (OR LINKED) CHOICES The condition that if British Petro is included in the portfolio then Trans-Texas Oil must also be included in the portfolio is an example of an *if–then* constraint. We can model this relationship as

$$B \leq T$$

or, if you prefer to have only a constant on the RHS,

$$B - T \leq 0$$

Note that if B equals 0 (i.e., British Petro is not included in the portfolio), this constraint allows T to equal either 0 or 1. However, if B equals 1, then T must also equal 1.

The relationship discussed here is a one-way linkage in that Trans-Texas Oil must be included if British Petro is included, but not vice versa. If the relationship is two way (i.e., either include both or include neither), we then rewrite the constraint as

$$B = T$$

or, once again if you prefer to have only a constant on the RHS,

$$B - T = 0$$

SOLVING THE PROBLEM The complete formulation of Simkin and Steinberg's problem is as follows:

$$\text{Maximize return} = \$50T + \$80B + \$90D + \$120H + \$110L + \$40S + \$75C$$

subject to the constraints

$\$480T + \$540B + \$680D + \$1{,}000H + \$700L + \$510S + \$900C$	$\leq \$3{,}000$	(investment limit)
$T + H + L$	≥ 2	(Texas companies)
$B + D$	≤ 1	(foreign companies)
$S + C$	$= 1$	(California companies)
B	$\leq T$	(Trans-Texas and British petro)
All variables	$= 0$ or 1	

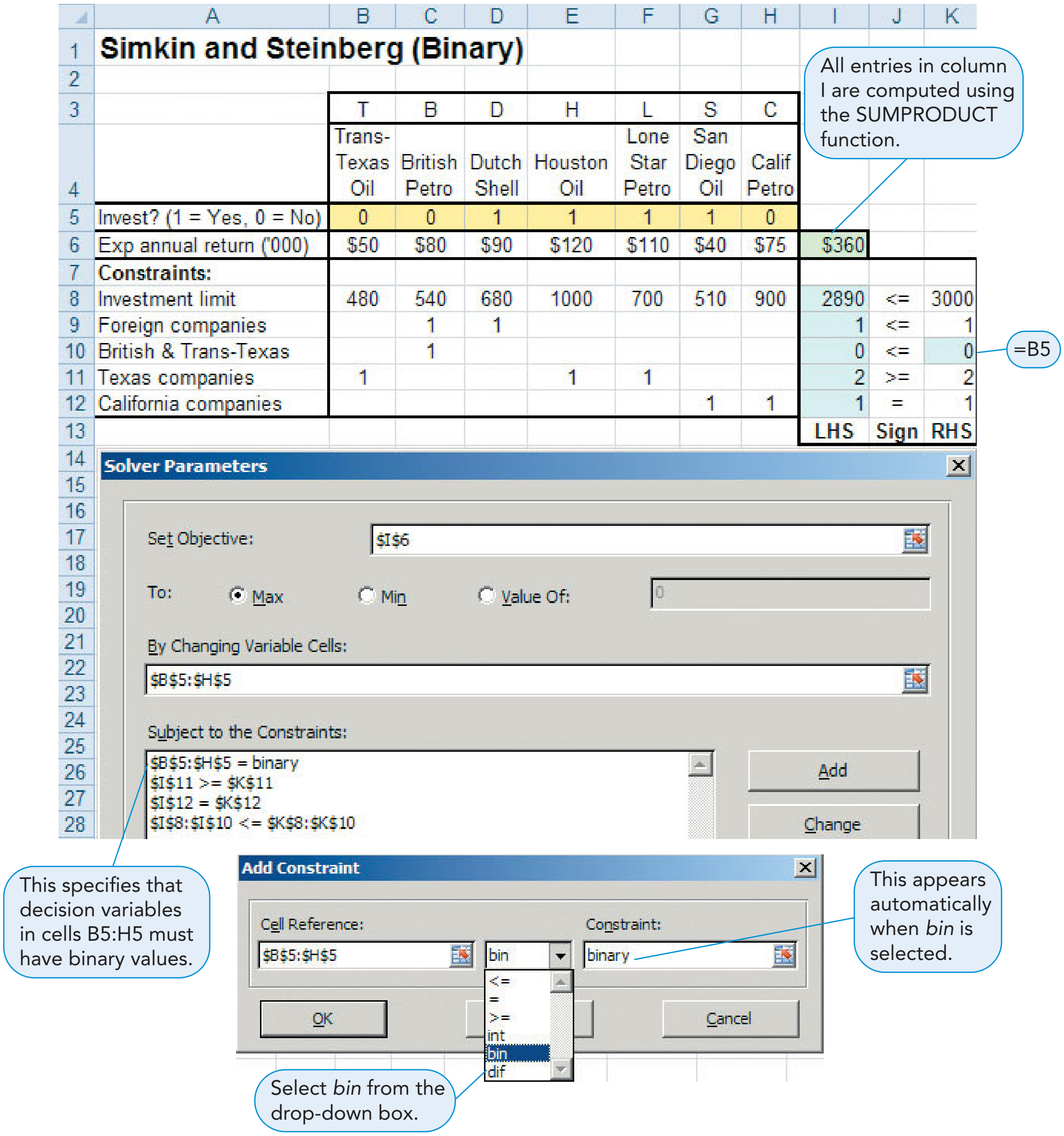
File: 6-3.xls

The Excel layout and Solver entries for Simkin and Steinberg's 0–1 problem are shown in Screenshot 6-3. The specification of the objective cell, changing variable cells, and constraint LHS and RHS cell references in the Solver Parameters window is similar to that used for LP and general IP models.

The binary requirement is specified as an additional constraint in Solver.

SPECIFYING THE BINARY REQUIREMENT To specify the binary requirement for all variables, we again use the Add option to include a new constraint. In the LHS entry for the new constraint (see Screenshot 6-3), we enter the cell reference for a decision variable that must

SCREENSHOT 6-3
Excel Layout and Solver Entries for Simkin and Steinberg—Binary IP



be binary valued. If there are several decision variables that must be binary valued, we can enter the entire cell range, provided that these variables are in contiguous cells. For Simkin and Steinberg's problem, we enter B5:H5 in this box, corresponding to binary variables *T* through *C*, respectively.

We then click the drop-down box in the Add Constraint window and click the choice bin. The word *binary* is automatically displayed in the box for the RHS entry. This indicates to Solver that all variables specified in the LHS box are binary variables.

INTERPRETING THE RESULTS Screenshot 6-3 shows that the optimal solution is for Simkin and Steinberg to recommend that the client invest in Dutch Shell (*D*), Houston Oil (*H*), Lone Star Petro (*L*), and San Diego Oil (*S*). The expected return is $360,000 (all values are in units of $1,000). Note that the solution invests only $2.89 million of the available $3 million. Why did this happen? There are two reasons for this: (1) Company stocks can only be bought in fixed blocks, and (2) specifications made by the client are possibly too restrictive.

Set Covering Problem at Sussex County

A set covering problem seeks to identify the optimal set of locations to cover a specified set of customers.

As noted earlier, set covering problems typically deal with trying to identify the optimal set of locations to cover or serve a specified set of customers. Consider the case of Sussex County, which needs to build health care clinics to serve seven communities (named A to G) in the region. Each clinic can serve communities within a maximum radius of 30 minutes' driving time, and a community may be served by more than one clinic. Table 6.3 shows the times it takes to travel between the seven communities. What is the minimum number of clinics that would be needed, and in which communities should they be located?

FORMULATING AND SOLVING THE PROBLEM The decision with regard to each community has to be one of two choices—either locate a clinic in that community or not. To formulate this problem, let us therefore associate a binary variable with each of the seven communities. For example, we define a binary variable, A, for community A as follows:

$$\begin{aligned} A &= 1 \text{ if a clinic is located at community A} \\ &= 0 \text{ if a clinic is not located at community A} \end{aligned}$$

In a similar manner, we define binary variables B, C, D, E, F, and G for communities B to G, respectively. Because Sussex County would like to minimize the number of clinics needed, we write the objective function as

$$\text{Minimize total number of clinics} = A + B + C + D + E + F + G$$

We now need to identify which communities are served by a clinic at a given location. For example, a clinic located at community A would serve communities A, B, and C because all three of these are within the 30-minute driving time limit. Table 6.4 shows the communities covered by the clinics at all seven locations.

TABLE 6.3
Sussex County Driving Times

	TO						
FROM	A	B	C	D	E	F	G
A	0	15	20	35	35	45	40
B	15	0	35	20	35	40	40
C	20	35	0	15	50	45	30
D	35	20	15	0	35	20	20
E	35	35	50	35	0	15	40
F	45	40	45	20	15	0	35
G	40	40	30	20	40	35	0

TABLE 6.4
Sussex County Community Coverage

COMMUNITY	COMMUNITIES WITHIN 30 MINUTES
A	A, B, C
B	A, B, D
C	A, C, D, G
D	B, C, D, F, G
E	E, F
F	D, E, F
G	C, D, G

The constraints need to ensure that each community is served (or covered) by at least one clinic. For this reason, they are called *covering constraints*. For Sussex County's problem, these constraints are as follows:

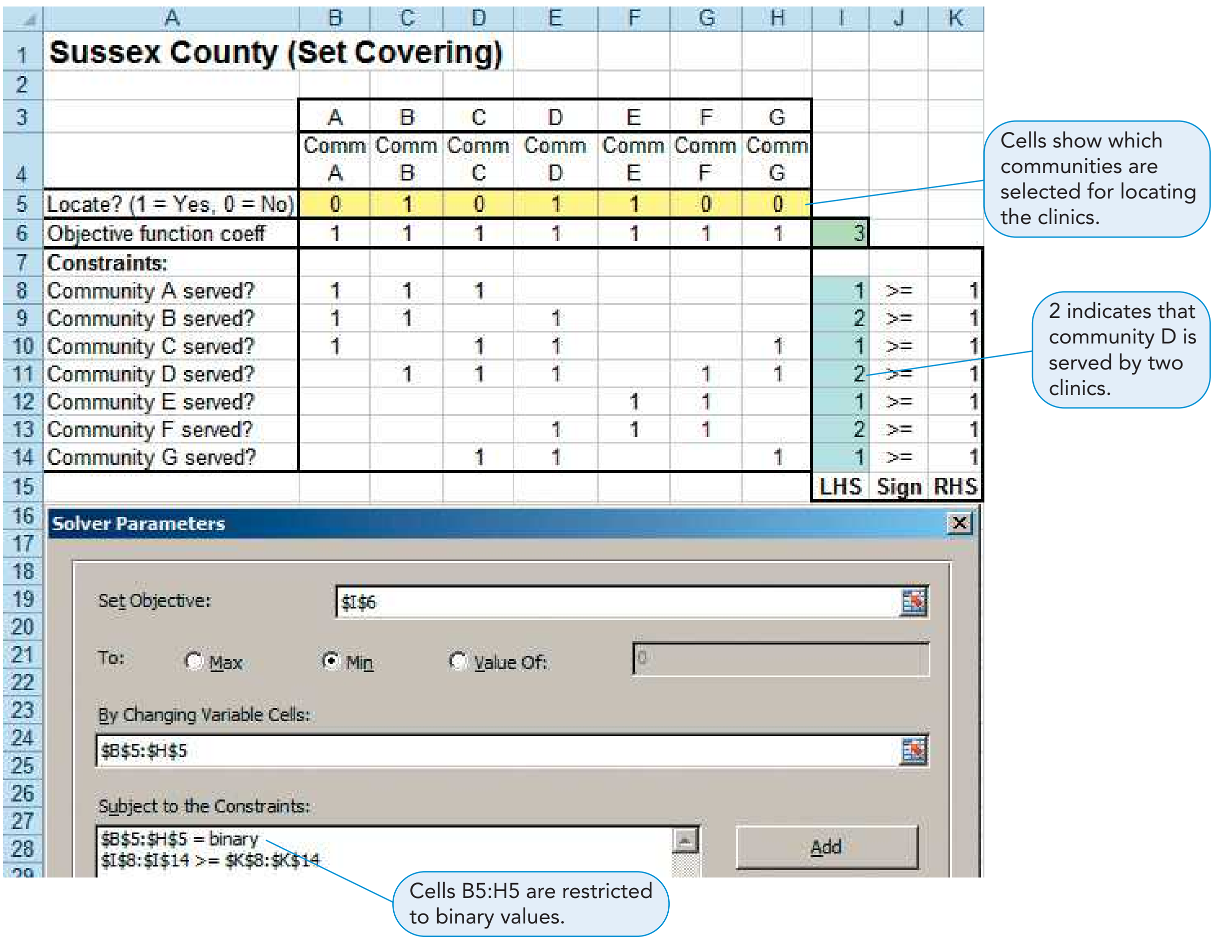
$$
\begin{array}{lll}
A + B + C & \geq 1 & \text{(community A is covered)} \\
A + B + D & \geq 1 & \text{(community B is covered)} \\
A + C + D + G & \geq 1 & \text{(community C is covered)} \\
B + C + D + F + G & \geq 1 & \text{(community D is covered)} \\
E + F & \geq 1 & \text{(community E is covered)} \\
D + E + F & \geq 1 & \text{(community F is covered)} \\
C + D + G & \geq 1 & \text{(community G is covered)} \\
\text{All variables} & = 0 \text{ or } 1 &
\end{array}
$$

There are no other constraints in Sussex County's problem. However, if necessary, we could use the three types of constraints described earlier—k out of n choices, mutually exclusive, and if–then—in Simkin and Steinberg's portfolio selection problem, to model any other specifications. For example, if Sussex does not want to locate clinics at both B and D, this would be modeled as a mutually exclusive constraint (i.e., $B + D \leq 1$).

WWW
File: 6-4.xls

The Excel layout and Solver entries for Sussex County's set covering problem are shown in Screenshot 6-4.

SCREENSHOT 6-4 Excel Layout and Solver Entries for Sussex County—Set Covering



	A	B	C	D	E	F	G	H	I	J	K
1	Sussex County (Set Covering)										
2											
3		A	B	C	D	E	F	G			
4		Comm A	Comm B	Comm C	Comm D	Comm E	Comm F	Comm G			
5	Locate? (1 = Yes, 0 = No)	0	1	0	1	1	0	0			
6	Objective function coeff	1	1	1	1	1	1	1	3		
7	Constraints:										
8	Community A served?	1	1	1					1	>=	1
9	Community B served?	1	1		1				2	>=	1
10	Community C served?	1		1	1			1	1	>=	1
11	Community D served?		1	1	1		1	1	2	>=	1
12	Community E served?					1	1		1	>=	1
13	Community F served?				1	1	1		2	>=	1
14	Community G served?			1	1			1	1	>=	1
15									LHS	Sign	RHS

IN ACTION Binary Integer Programming Facilitates Better Course Scheduling at the Universidad de Chile

In its simplest form, the course and examination scheduling problem can be defined as the assignment of a set of courses to different time slots and classrooms while satisfying certain requirements. These requirements can vary widely based on factors such as the institution's policies, room availabilities, level of classes being scheduled, etc.

The Executive Education Unit (EEU) of the Universidad de Chile offers courses primarily for professionals and high-level executives. Ensuring proper schedules for this high-profile audience is critical because any perception of disorganization would affect the EEU adversely. Between 2003 and 2008, about 7,000 students attended EEU courses.

The eClasSkeduler decision support system used at EEU consists of four modules: (1) input information module that stores all information relating to courses, classrooms, and instructors, (2) user interface module that transforms the input data to the format necessary for the binary integer programming (BIP) optimization model, (3) optimization module that contains the source code for the BIP model, and (4) report module that transforms the BIP model's results to an user-friendly management report with various performance indicators.

The use of eClasSkeduler has benefited all EEU participants by curtailing operating costs, lowering unused classroom capacity, and producing fewer schedule conflicts and off-premise classroom assignments.

Sources: Based on J. Miranda. "eClasSkeduler: A Course Scheduling System for the Executive Education Unit at the Universidad de Chile," *Interfaces* 40, 3 (May–June 2010): 196–207.

INTERPRETING THE RESULTS The results indicate that Sussex County will need to open three clinics, one each at communities B, D, and E, to serve the seven communities. Residents of three of the seven communities (B, D, and F) will be served by two clinics each, while residents of the other four communities will be served by only one clinic each.

Note that because Sussex County permits more than one clinic to serve a community, the constraints in our model included the $\geq$ sign. In contrast, if Sussex County wants each community to be served by exactly one clinic, the constraints would include the $=$ sign. This specification may, however, cause the model to be infeasible in some cases. For example, the driving times could be such that it would be impossible to find a set of locations that uniquely serve all communities. In fact, this is the case in Sussex County's problem (see if you can verify this).

In general, there is no easy way to detect the presence of alternate optimal solutions for IP models.

ALTERNATE OPTIMAL SOLUTIONS It turns out that there are multiple optimal solutions to Sussex County's model. For example, locating clinics in communities A, C, and F is also optimal (see if you can verify this solution). Recall that in Chapter 4 we studied how to use the Solver Sensitivity Report to detect the presence of alternate optimal solutions. However, because we cannot obtain Sensitivity Reports for IP models, we cannot adopt that strategy here. In general, there is no easy way to detect the presence of alternate optimal solutions for IP models.

6.4 Mixed Integer Models: Fixed-Charge Problems

In all LP and general integer models studied so far, we typically deal with situations in which the total cost is directly proportional to the magnitude of the decision variable. For example, if X denotes the number of toasters we will be making, and if each toaster costs \$10 to make, the total cost of making toasters is written as \$10X. Such costs per unit are referred to as *variable* costs.

In many situations, however, there are fixed costs in addition to the per-unit variable costs. These costs may include the costs to set up machines for the production run, construction costs to build a new facility, or design costs to develop a new product. Unlike variable costs, these fixed costs are independent of the volume of production. They are incurred whenever the decision to go ahead with a project or production run is made.

***Fixed-charge problems** include fixed costs in addition to variable costs.*

Problems that involve both fixed and variable costs are a classic example of **mixed integer programming** models. We call such problems **fixed-charge problems**.

We use binary variables to model the fixed cost issue (e.g., whether we will incur the setup cost or not). Either linear or integer variables can be used to deal with the variable costs issue, depending on the nature of these variables. In formulating the model, we need to ensure that whenever the decision variable associated with the variable cost is nonzero, the binary variable associated with the fixed cost takes on a value of 1 (i.e., the fixed cost is also incurred).

To illustrate this type of situation, let us revisit the Hardgrave Machine Company facility location example that we first studied as Solved Problem 5-1 in Chapter 5 (see page 193).

Locating a New Factory for Hardgrave Machine Company

Hardgrave Machine Company produces computer components at its factories in Cincinnati, Kansas City, and Pittsburgh. These factories have not been able to keep up with demand for orders at Hardgrave's four warehouses in Detroit, Houston, New York, and Los Angeles. As a result, the firm has decided to build a new factory to expand its productive capacity. The two sites being considered are Seattle, Washington, and Birmingham, Alabama. Both cities are attractive in terms of labor supply, municipal services, and ease of factory financing.

Table 6.5 presents the production costs and monthly supplies at each of the three existing factories, monthly demands at each of the four warehouses, and estimated production costs at the two proposed factories. Transportation costs from each factory to each warehouse are summarized in Table 6.6.

Sunk costs are not considered in the optimization model.

In addition to this information, Hardgrave estimates that the monthly fixed cost of operating the proposed facility in Seattle would be $400,000. The Birmingham plant would be somewhat cheaper, due to the lower cost of living at that location. Hardgrave therefore estimates that the monthly fixed cost of operating the proposed facility in Birmingham would be $325,000. Note that the fixed costs at *existing* plants need not be considered here because they will be incurred regardless of which new plant Hardgrave decides to open—that is, they are sunk costs.

As in Chapter 5, the question facing Hardgrave is this: Which of the new locations, in combination with the existing plants and warehouses, will yield the lowest cost? Note that the

TABLE 6.5
Hardgrave Machine's Demand and Supply Data

WAREHOUSE	MONTHLY DEMAND (UNITS)	PRODUCTION PLANT	MONTHLY SUPPLY	COST TO PRODUCE ONE UNIT
Detroit	10,000	Cincinnati	15,000	$48
Houston	12,000	Kansas City	6,000	$50
New York	15,000	Pittsburgh	14,000	$52
Los Angeles	9,000		35,000	
	46,000			

Supply needed from new plant = 46,000 − 35,000 = 11,000 units per month

ESTIMATED PRODUCTION COST PER UNIT AT PROPOSED PLANTS	
Seattle	$53
Birmingham	$49

TABLE 6.6
Hardgrave Machine's Shipping Costs

	TO			
FROM	DETROIT	HOUSTON	NEW YORK	LOS ANGELES
Cincinnati	$25	$55	$40	$60
Kansas City	$35	$30	$50	$40
Pittsburgh	$36	$45	$26	$66
Seattle	$60	$38	$65	$27
Birmingham	$35	$30	$41	$50

unit cost of shipping from each plant to each warehouse is found by adding the shipping costs (Table 6.6) to the corresponding production costs (Table 6.5). In addition, the solution needs to consider the monthly fixed costs of operating the new facility.

Recall that we handled this problem in Solved Problem 5-1 by setting up and solving two separate transportation models—one for each of the two new locations. In the following pages, we show how we can use binary variables to model Hardgrave's problem as a single mixed, binary integer programming model.

DECISION VARIABLES There are two types of decisions to be made in this problem. The first involves deciding which of the new locations (Seattle or Birmingham) to select for the new plant. The second involves trying to decide the shipment quantities from each plant (including the new plant) to each of the warehouses.

We use binary variables to model the opening of a plant.

To model the first decision, we associate a binary variable with each of the two locations. Let

$$\begin{aligned} Y_S &= 1 \text{ if Seattle is selected for the new plant} \\ &= 0 \text{ otherwise} \\ Y_B &= 1 \text{ if Birmingham is selected for the new plant} \\ &= 0 \text{ otherwise} \end{aligned}$$

We use regular variables (continuous-valued or general integer) to model the shipping quantities.

To model the shipping quantities, we once again use double-subscripted variables, as discussed in Chapter 5. Note that there are 5 plants (3 existing and 2 proposed) and 4 warehouses in the problem. Therefore, the model will include 20 decision variables denoting the shipping quantities (one variable for each possible shipping route). Let

$$X_{ij} = \text{Number of units shipped from plant } i \text{ to warehouse } j$$

where

$$\begin{aligned} i &= \text{C (Cincinnati), K (Kansas City), P (Pittsburgh), S (Seattle), or } B \text{ (Birmingham)} \\ j &= \text{D (Detroit), H (Houston), N (New York), or L (Los Angeles)} \end{aligned}$$

OBJECTIVE FUNCTION Let us first model the objective function. We want to minimize the total cost of producing and shipping the components and the monthly fixed costs of maintaining the new facility. This can be written as

$$\begin{aligned} \text{Minimize total costs} = {} & \$73X_{CD} + \$103X_{CH} + \$88X_{CN} + \$108X_{CL} \\ & + \$85X_{KD} + \$80X_{KH} + \$100X_{KN} + \$90X_{KL} \\ & + \$88X_{PD} + \$97X_{PH} + \$78X_{PN} + \$118X_{PL} \\ & + \$113X_{SD} + \$91X_{SH} + \$118X_{SN} + \$80X_{SL} \\ & + \$84X_{BD} + \$79X_{BH} + \$90X_{BN} + \$99X_{BL} \\ & + \$400{,}000Y_S + \$325{,}000Y_B \end{aligned}$$

The last two terms in the expression for the objective function represent the fixed costs. Note that these costs will be incurred only if the plant is built at the location (i.e., the variable Y_S or Y_B has a value of 1).

CONSTRAINTS We need to write flow balance constraints for each of the plants and warehouses. Recall that at each node, the flow balance constraint ensures that

$$\text{Net flow} = (\text{Total flow } in \text{ to node}) - (\text{Total flow } out \text{ of node})$$

At source nodes, the net flow is a negative quantity and represents the amount of goods (flow) created at that node. In contrast, at destination nodes, the net flow is a positive quantity and represents the amount of goods (flow) consumed at that node. Because this is a balanced problem, all flow balance constraints can be written as equalities.

The flow balance constraints at the existing plants (Cincinnati, Kansas City, and Pittsburgh) are straightforward and can be written as

$$\begin{aligned} (0) - (X_{CD} + X_{CH} + X_{CN} + X_{CL}) &= -15{,}000 && \text{(Cincinnati supply)} \\ (0) - (X_{KD} + X_{KH} + X_{KN} + X_{KL}) &= -\ 6{,}000 && \text{(Kansas City supply)} \\ (0) - (X_{PD} + X_{PH} + X_{PN} + X_{PL}) &= -14{,}000 && \text{(Pittsburgh supply)} \end{aligned}$$

Supply is available at a plant only if the plant is opened.

However, when writing the flow balance constraint for a new plant (Seattle or Birmingham), we need to ensure that a supply is available at that plant *only* if the plant is actually built. For example, the supply at Seattle is 11,000 units if the new plant is built there and 0 otherwise. We can model this as follows:

$$(0) - (X_{SD} + X_{SH} + X_{SN} + X_{SL}) = -11{,}000Y_S \qquad \text{(Seattle supply)}$$
$$(0) - (X_{BD} + X_{BH} + X_{BN} + X_{BL}) = -11{,}000Y_B \qquad \text{(Birmingham supply)}$$

Note that if Seattle is selected for the new plant, Y_S equals 1. Hence, a supply of 11,000 is available there. In contrast, if Seattle is not selected for the new plant, Y_S equals 0. Hence, the supply in the flow balance constraint becomes 0; that is, all flows from Seattle have to equal 0. The flow balance constraint for Birmingham works in a similar manner.

The flow balance constraints at the four existing warehouses (Detroit, Houston, New York, and Los Angeles) can be written as

$$X_{CD} + X_{KD} + X_{PD} + X_{SD} + X_{BD} = 10{,}000 \qquad \text{(Detroit supply)}$$
$$X_{CH} + X_{KH} + X_{PH} + X_{SH} + X_{BH} = 12{,}000 \qquad \text{(Houston supply)}$$
$$X_{CN} + X_{KN} + X_{PN} + X_{SN} + X_{BN} = 15{,}000 \qquad \text{(New York supply)}$$
$$X_{CL} + X_{KL} + X_{PL} + X_{SL} + X_{BL} = 9{,}000 \qquad \text{(Los Angeles supply)}$$

Only one of the two sites can be selected.

Finally, we need to ensure that exactly one of the two sites is selected for the new plant. This is another example of the mutually exclusive variables discussed in section 6.3. We can express this as

$$Y_S + Y_B = 1$$

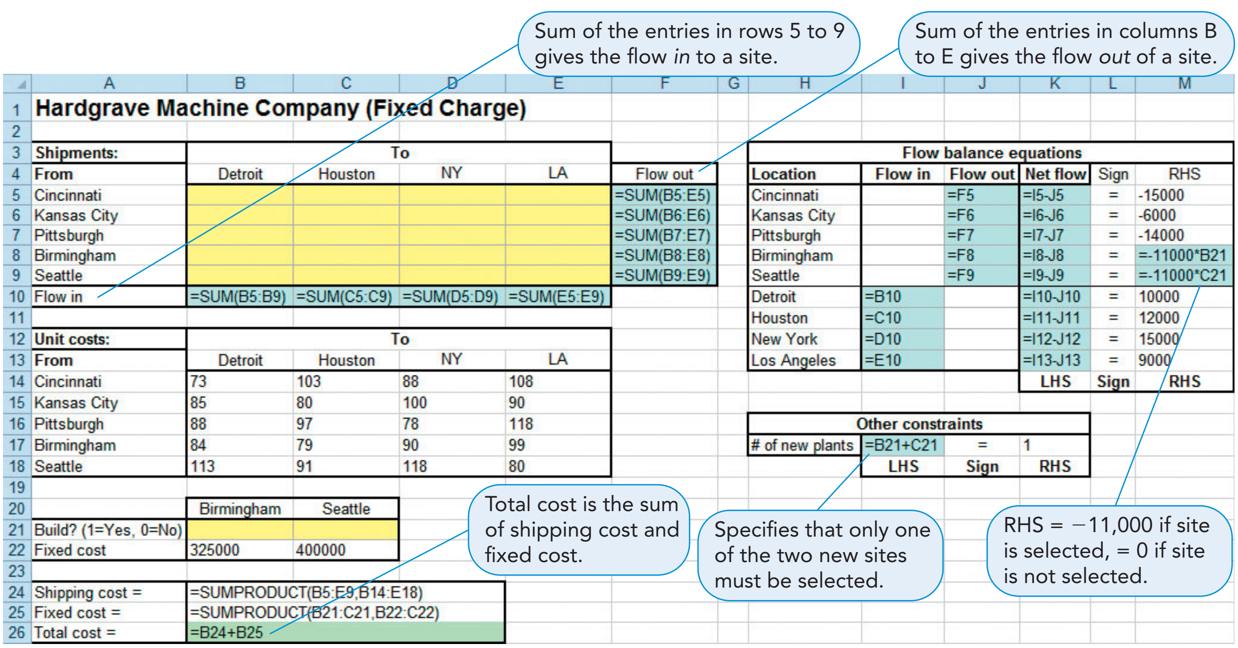
File: 6-5.xls

SOLVING THE PROBLEM AND INTERPRETING THE RESULTS The formula view of the Excel layout for Hardgrave's fixed-charge problem is shown in Screenshot 6-5A. The Solver entries and optimal solution are shown in Screenshot 6-5B.

Referring to Solved Problem 5-1 on page 193, we see that the cost of shipping was \$3,704,000 if the new plant was built in Seattle. This cost was \$3,741,000 if the new plant was built in Birmingham. With the fixed costs included, these costs would be

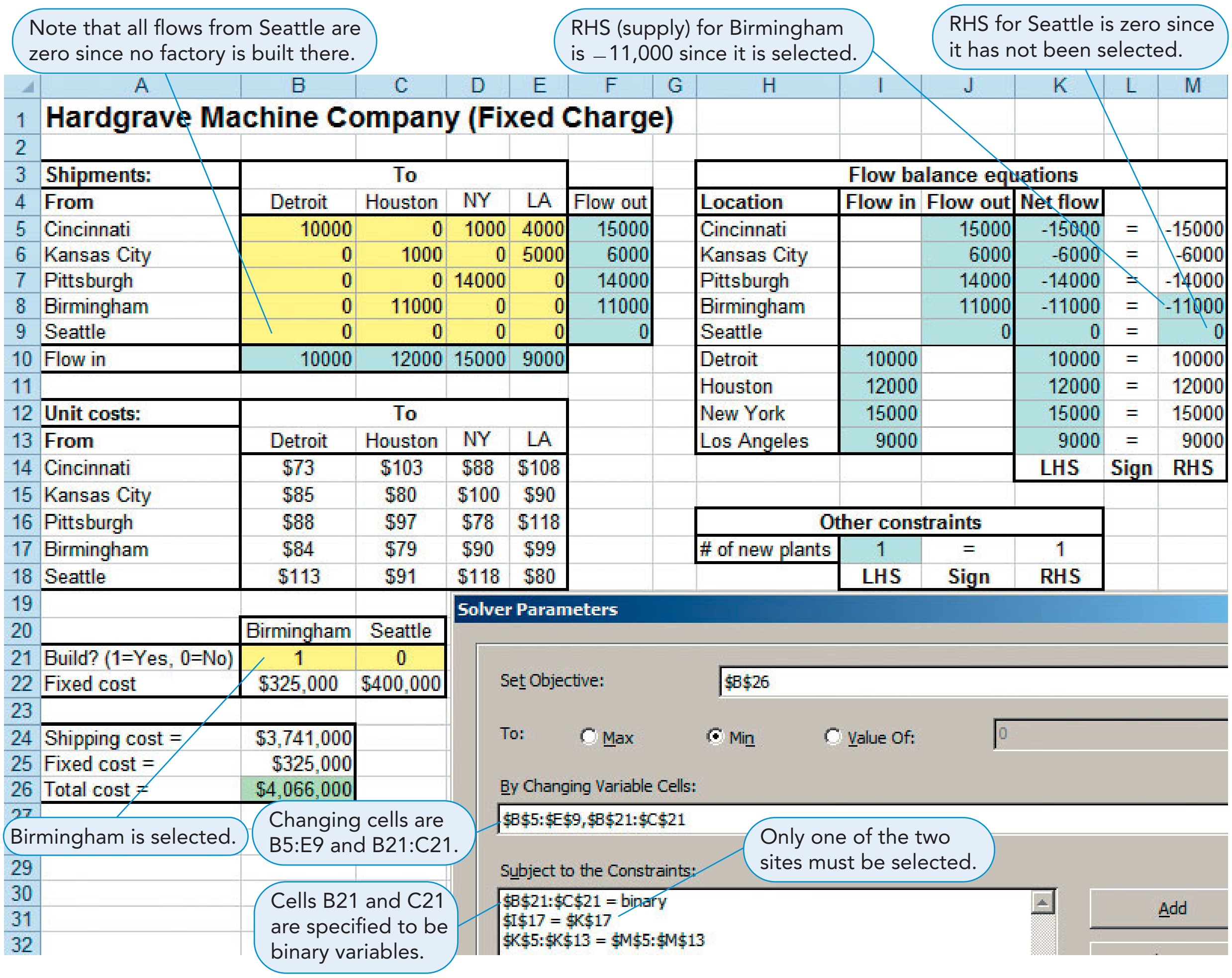
Seattle:	\$3,704,000 + \$400,000 = \$4,104,000
Birmingham:	\$3,741,000 + \$325,000 = \$4,066,000

SCREENSHOT 6-5A **Formula View of Excel Layout for Hardgrave Machine—Fixed Charge**



	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Hardgrave Machine Company (Fixed Charge)												
2													
3	Shipments:		To					Flow balance equations					
4	From	Detroit	Houston	NY	LA	Flow out		Location	Flow in	Flow out	Net flow	Sign	RHS
5	Cincinnati					=SUM(B5:E5)		Cincinnati		=F5	=I5-J5	=	-15000
6	Kansas City					=SUM(B6:E6)		Kansas City		=F6	=I6-J6	=	-6000
7	Pittsburgh					=SUM(B7:E7)		Pittsburgh		=F7	=I7-J7	=	-14000
8	Birmingham					=SUM(B8:E8)		Birmingham		=F8	=I8-J8	=	=-11000*B21
9	Seattle					=SUM(B9:E9)		Seattle		=F9	=I9-J9	=	=-11000*C21
10	Flow in	=SUM(B5:B9)	=SUM(C5:C9)	=SUM(D5:D9)	=SUM(E5:E9)			Detroit	=B10		=I10-J10	=	10000
11								Houston	=C10		=I11-J11	=	12000
12	Unit costs:		To					New York	=D10		=I12-J12	=	15000
13	From	Detroit	Houston	NY	LA			Los Angeles	=E10		=I13-J13	=	9000
14	Cincinnati	73	103	88	108						LHS	Sign	RHS
15	Kansas City	85	80	100	90								
16	Pittsburgh	88	97	78	118			Other constraints					
17	Birmingham	84	79	90	99			# of new plants	=B21+C21	=	1		
18	Seattle	113	91	118	80				LHS	Sign	RHS		
19													
20		Birmingham	Seattle										
21	Build? (1=Yes, 0=No)												
22	Fixed cost	325000	400000										
23													
24	Shipping cost =	=SUMPRODUCT(B5:E9,B14:E18)											
25	Fixed cost =	=SUMPRODUCT(B21:C21,B22:C22)											
26	Total cost =	=B24+B25											

SCREENSHOT 6-5B Solver Entries and Solution for Hardgrave Machine—Fixed Charge



Total cost includes fixed costs and shipping costs.

That is, Hardgrave should select Birmingham as the site for the new plant. Screenshot 6-5B shows this solution. Note that the shipping quantities in this solution are the same as those obtained in Solved Problem 5-1 for the solution with the Birmingham plant (as shown in Screenshot 5-9B on page 194).

6.5 Goal Programming Models

Goal programming permits multiple objectives.

In today's business environment, maximizing profit (or minimizing cost) is not always the only objective that a firm may specify. In many cases, maximizing profits is just one of several objectives that may include maximizing machine utilization, maintaining full employment, providing quality ecological management, minimizing noise level in the neighborhood, and meeting numerous other non-economic targets. Often, some of these objectives are conflicting (i.e., it may not be possible to simultaneously achieve these objectives).

Mathematical programming techniques such as LP and IP have the shortcoming that their objective function can deal with only a single criterion, such as profit, cost, or some such measure. To overcome this shortcoming, an important technique that has been developed to handle decision models involving multiple objectives is called **goal programming (GP)**. This technique began with the work of Charnes and Cooper in 1961 and was refined and extended by Ignizio in the 1970s.[3]

[3] Charnes, A., and W. W. Cooper. *Management Models and Industrial Applications of Linear Programming*. New York: Wiley, 1961.
Ignizio, J. P. *Goal Programming and Extensions*. Lexington, MA: D.C. Heath and Company, 1976.

Whereas LP optimizes, GP satisfices.

How do LP/IP and GP models differ? In LP/IP models, we try to find the best possible value for a single objective. That is, the aim is to *optimize* a single measure. In GP models, on the other hand, we first set a goal (or desired target) for each objective. In most decision modeling situations, some of these goals may be achievable only at the expense of other goals. We therefore establish a hierarchy or rank of importance among these goals so that lower-ranked goals are given less prominence than higher-ranked goals. Based on this hierarchy, GP then attempts to reach a "satisfactory" level for each goal. That is, GP tries to **satisfice** the multiple objectives (i.e., come as close as possible to their respective goals) rather than optimize them. Nobel laureate Herbert A. Simon of Carnegie-Mellon University states that modern managers may not be able to optimize but may instead have to satisfice to reach goals.

In GP we want to minimize deviation variables, which are the only terms in the objective function.

How does GP satisfice the goals? Instead of trying to maximize or minimize the objective functions directly, as in LP/IP, with GP we try to minimize *deviations* between the specified goals and what we can actually achieve for the multiple objective functions within the given constraints. Deviations can be either positive or negative, depending on whether we overachieve or underachieve a specific goal. These deviations are not only real decision variables in the GP model, but they are also the only terms in the objective function. The objective is to minimize some function of these **deviation variables**.

Goal Programming Example: Wilson Doors Company

To illustrate the formulation of a GP problem, let us consider the product mix problem faced by the Wilson Doors Company. The company manufactures three styles of doors—exterior, interior, and commercial. Each door requires a certain amount of steel and two separate production steps: forming and assembly. Table 6.7 shows the material requirement, forming and assembly times, and selling price per unit of each product, along with the monthly availability of all resources.

FORMULATING AND SOLVING THE LP MODEL Let us denote E = number of exterior doors to make, I = number of interior doors to make, and C = number of commercial doors to make. If Wilson's management had just a single objective (i.e., to maximize total sales), the LP formulation for the problem would be written as

$$\text{Maximize total sales} = \$70E + \$110I + \$110C$$

subject to the constraints

$$\begin{aligned} 4E + 3I + 7C &\le 9{,}000 && \text{(steel usage)} \\ 2E + 4I + 3C &\le 6{,}000 && \text{(forming time)} \\ 2E + 3I + 4C &\le 5{,}200 && \text{(assembly time)} \\ E, I, C &\le 0 \end{aligned}$$

File: 6-6.xls, sheet: 6-6 LP

The optimal LP solution turns out to be $E = 1{,}400$, $I = 800$, and $C = 0$, for a total sales of \$186,000. At this stage, you should be able to easily verify this yourself. However, for your convenience, this LP solution is included in the Excel file *6-6.xls* on the Companion Website for this textbook; see the worksheet named *6-6 LP*.

SPECIFYING THE GOALS Now suppose that Wilson is not happy with this LP solution because it generates no sales from commercial doors. In contrast, exterior doors generate

TABLE 6.7
Data for Wilson Doors

	EXTERIOR	INTERIOR	COMMERCIAL	AVAILABILITY
Steel (lb./door)	4	3	7	9,000 pounds
Forming (hr./door)	2	4	3	6,000 hours
Assembly (hr./door)	2	3	4	5,200 hours
Selling price/door	\$70	\$110	\$110	

IN ACTION The Use of Goal Programming for TB Drug Allocation in Manila

Allocation of resources is critical when applied to the health industry. It is a matter of life and death when neither the right supply nor the correct quantity is available to meet patient demand. This was the case faced by the Manila (Philippines) Health Center, whose drug supply to patients afflicted with Category 1 tuberculosis (TB) was not being efficiently allocated to its 45 regional health centers. When the TB drug supply does not reach patients on time, the disease becomes worse and can result in death. Only 74% of TB patients were being cured in Manila, 11% short of the 85% target cure rate set by the government. Unlike other diseases, TB can be treated only with four medicines and cannot be cured by alternative drugs.

Researchers at the Mapka Institute of Technology set out to create a model, using GP, to optimize the allocation of resources for TB treatment while considering supply constraints. The objective function of the model was to meet the target cure rate of 85% (which is the equivalent of minimizing the underachievement in the allocation of anti-TB drugs to the 45 centers). Four goal constraints considered the interrelationships among variables in the distribution system. Goal 1 was to satisfy the medication requirement (a six-month regimen) for each patient. Goal 2 was to supply each health center with the proper allocation. Goal 3 was to satisfy the cure rate of 85%. Goal 4 was to satisfy the drug requirements of each health center.

The GP model successfully dealt with all these goals and raised the TB cure rate to 88%, a 13% improvement in drug allocation over the previous distribution approach. This means that 335 lives per year were saved through this thoughtful use of GP.

Source: Based on G. J. C. Esmeria. "An Application of Goal Programming in the Allocation of Anti-TB Drugs in Rural Health Centers in the Philippines," *Proceedings of the 12th Annual Conference of the Production and Operations Management Society* (March 2001), Orlando, FL.

98,000 (=70 × 1,400) and interior doors generate 88,000 (=110 × 800) in sales. This would imply that while the sales agents for exterior and interior doors get sales bonuses this month, the sales agent for commercial doors gets nothing. To alleviate this situation, Wilson would prefer that each type of door contribute a certain level of sales. Wilson is, however, not willing to compromise too much on the *total* sales. Further, it does not want to be unduly unfair to the sales agents for exterior and interior doors by taking away too much of their sales potential (and hence, their sales bonus). Considering all issues, suppose Wilson sets the following goals:

Goal 1: Achieve total sales of at least \$180,000

Goal 2: Achieve exterior doors sales of at least \$70,000

Goal 3: Achieve interior doors sales of at least \$60,000

Goal 4: Achieve commercial doors sales of at least \$35,000

Goals look similar to constraints except that goals may remain unsatisfied in the final solution.

Notice that these goals look somewhat similar to constraints. However, there is a key difference. Constraints are restrictions that *must* be satisfied by the solution. Goals, on the other hand, are specifications that we would *like* to satisfy. However, it is acceptable to leave one or more goals unsatisfied in the final solution if it is impossible to satisfy them (because of other, possibly conflicting, goals and constraints in the model). We now have a GP problem in which we want to find the product mix that achieves these four goals as much as possible, given the production resource constraints.

We must first define two deviation variables for each goal in a GP problem.

FORMULATING THE GP MODEL To formulate any problem as a GP problem, we must first define two deviation variables for each goal. These two deviation variables represent, respectively, the extent to which a goal is underachieved or overachieved. Because there are four goals in Wilson's problem, we define eight deviation variables, as follows:

d_T^- = amount by which the total sales goal is underachieved

d_T^+ = amount by which the total sales goal is overachieved

d_E^- = amount by which the exterior doors sales goal is underachieved

d_E^+ = amount by which the exterior doors sales goal is overachieved

d_I^- = amount by which the interior doors sales goal is underachieved

d_I^+ = amount by which the interior doors sales goal is overachieved

$d_C^- =$ amount by which the commercial doors sales goal is underachieved

$d_C^+ =$ amount by which the commercial doors sales goal is overachieved

We use the deviation variables to express goals as equations.

Using these deviation variables, we express the four goals mathematically as follows:

$$
\begin{array}{llll}
70E + 110I + 110C + d_T^- - d_T^+ & = & 180{,}000 & \text{(total sales goal)} \\
70E + d_E^- - d_E^+ & = & 70{,}000 & \text{(exterior doors sales goal)} \\
110I + d_I^- - d_I^+ & = & 60{,}000 & \text{(interior doors sales goal)} \\
110C + d_C^- - d_C^+ & = & 35{,}000 & \text{(commercial doors sales goal)}
\end{array}
$$

The first equation states that the total sales (i.e., $\$70E + \$110I + \$110C$) plus any underachievement of total sales minus any overachievement of total sales has to equal the goal of \$180,000. For example, the LP solution ($E = 1{,}400$, $I = 800$, and $C = 0$) yields total sales of \$186,000. Because this exceeds the goal of \$180,000 by \$6,000, d_{T}^+ would equal \$6,000, and d_{T}^{--} would equal \$0. Note that it is not possible for both d_{T}^+ and d_{T}^- to be nonzero at the same time because it is not logical for a goal to be both underachieved and overachieved at the same time. The second, third, and fourth equations specify a similar issue with regard to sales from exterior, interior, and commercial doors, respectively.

We are concerned only about minimizing the underachievement of goals here.

Because all four of Wilson's goals specify that their targets should be *at least* met, we want to minimize only the level of underachievement in each goal. That is, we are not concerned if any or all goals are overachieved. With this background information, we can now formulate Wilson's problem as a single GP model, as follows:

$$\text{Minimize total underachievement of goals} = d_T^- + d_E^- + d_I^- + d_C^+$$

subject to the constraints

$$
\begin{array}{llll}
70E + 110I + 110C + d_T^- + d_T^+ & = & 180{,}000 & \text{(total sales goal)} \\
70E + d_E^- + d_E^+ & = & 70{,}000 & \text{(exterior doors sales goal)} \\
110I + d_I^- + d_I^+ & = & 60{,}000 & \text{(interior doors sales goal)} \\
110C + d_C^- + d_C^+ & = & 35{,}000 & \text{(commercial doors sales goal)} \\
4E + 3I + 7C & \le & 9{,}000 & \text{(steel usage)} \\
2E + 4I + 3C & \le & 6{,}000 & \text{(forming time)} \\
2E + 3I + 4C & \le & 5{,}200 & \text{(assembly time)} \\
\multicolumn{4}{l}{E, I, C, d_T^-, d_T^+, d_E^-, d_E^+, d_I^-, d_I^+, d_C^-, d_C^+ \ge 0}
\end{array}
$$

Deviation variables are 0 if a goal is fully satisfied.

If Wilson were just interested in *exactly* achieving all four goals, how would the objective function change? In that case, we would specify it to minimize the total underachievement and overachievement (i.e., the sum of all eight deviation variables). This, of course, is probably not a reasonable objective in practice because Wilson is not likely to be upset with an overachievement of any of its sales goals.

In general, once all the goals have been defined in a GP problem, management should analyze each goal to see if it wishes to include only one or both of the deviation variables for that goal in the minimization objective function. In some cases, the goals could even be one-sided in that it is not even feasible for one of the deviation variables to be nonzero. For example, if Wilson specifies that the \$180,000 target for total sales is an absolute minimum (i.e., it cannot be violated), the underachievement deviation variable d_{T}^- can be completely eliminated from the GP model.

There are approaches to solve GP models: using (1) **weighted goals** *and (2)* **ranked goals**.

Now that we have formulated Wilson's GP model with the four goals, how do we solve it? There are two approaches commonly used in practice: (1) using **weighted goals** and (2) using **ranked goals** (or prioritized goals). Let us now discuss each of these approaches.

Solving Goal Programming Models with Weighted Goals

Weights can be used to distinguish between different goals.

As currently formulated, Wilson's GP model assumes that all four goals are equally important to its managers. That is, because the objective function is just the sum of the four deviation variables (d_{T}^-, d_{E}^-, d_{I}^-, *and* d_{C}^-) a unit underachievement in the total sales goal (d_{T}^-) has the same impact on the objective function value as a unit underachievement in any of the other three sales goals (d_E^-, d_I^-, or d_C^-) If that is indeed the case in Wilson's problem, we can simply solve the model as currently formulated. However, as noted earlier, it is common in practice for managers to rank different goals in some hierarchical fashion.

FORMULATING THE WEIGHTED GP MODEL Suppose Wilson specifies that the total sales goal is five times as important as each of the other three sales goals. To include this specification in the weighted goal approach for solving GP models, we assign numeric weights to each deviation variable in the objective function. These weights serve as the objective coefficients for the deviation variables. The magnitude of the weight assigned to a specific deviation variable would depend on the relative importance of that goal. In Wilson's case, because minimizing d_T^- is five times as important as minimizing d_E^-, d_I^-, d_C^-, or d_C^-, we could assign the following weights to the four goals:

Goal 1: Achieve total sales of at least \$180,000	Weight = 5
Goal 2: Achieve exterior doors sales of at least \$70,000	Weight = 1
Goal 3: Achieve interior doors sales of at least \$60,000	Weight = 1
Goal 4: Achieve commercial doors sales of at least \$35,000	Weight = 1

With this information, we can now write the objective function with weighted goals for Wilson's model as

$$\text{Minimize total } \textit{weighted} \text{ underachievement of goals} = 5d_T^- + d_E^- + d_I^- + d_C^-$$

In the weighted goals approach, the problem reduces to an LP model with a single objective function.

The constraints are as listed earlier for the model. The problem now reduces to an LP model with a single objective function. Setting up this model on Excel and solving it by using Solver therefore become rather straightforward tasks.

File: 6-6.xls, sheet: 6-6A

SOLVING THE WEIGHTED GP MODEL The Excel layout and Solver entries for Wilson's problem with weighted goals are shown in Screenshot 6-6A. Note that the model includes 11 decision variables (3 product variables associated with the three types of doors and 8 deviation variables associated with the four goals). The results also show the extent to which each goal has been achieved (shown in cells P8:P11).

INTERPRETING THE RESULTS The optimal weighted GP solution is for Wilson to produce 1,000 exterior doors, 800 interior doors, and 200 commercial doors. This results in total revenue of \$180,000, which exactly satisfies that goal (i.e., d_T^+ and d_T^- are both equal to 0). Regarding the goals for the different types of doors, the exterior doors sales goal is also exactly satisfied, while the interior doors sales goal is overachieved by \$28,000. In contrast, sales from commercial doors is only \$22,000 ($= \$110 \times 200$), which underachieves the goal of \$35,000 by \$13,000. Wilson should, however, be willing to accept this result because it is more concerned about the total sales goal (and hence, assigned it a larger weight) than with the commercial doors sales goal. That is, in trying to satisfy Wilson's *stronger* desire to generate at least \$180,000 in total sales, the weighted GP solution continues to leave the commercial doors sales goal underachieved to a certain extent.

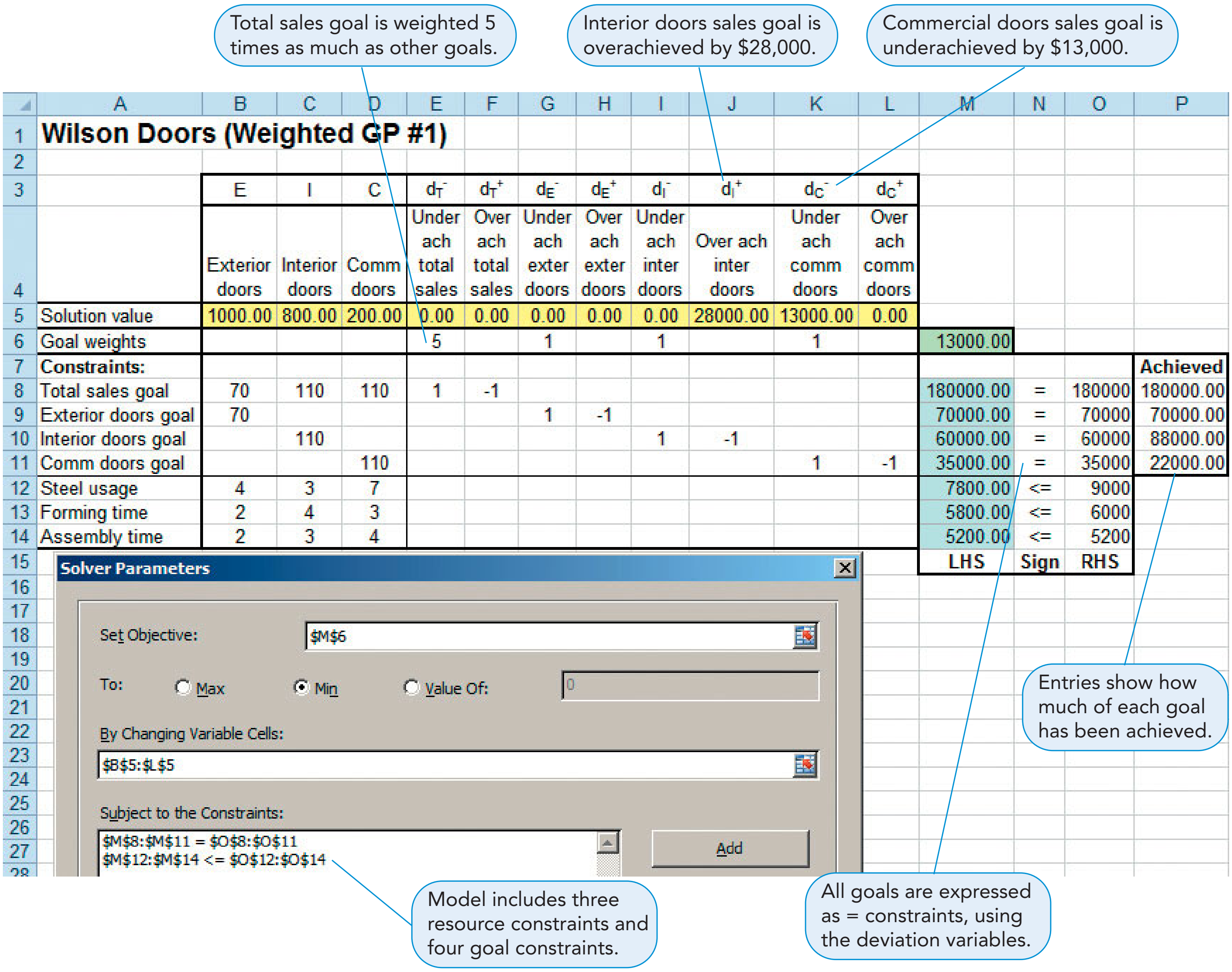
The weighted goals approach has two major drawbacks.

DRAWBACKS OF THE WEIGHTED GOALS APPROACH Although the weighted goals approach is rather easy to use, it suffers from two major drawbacks. First, it is appropriate to use only if all the goals (and hence, the deviation variables) are being measured in the same units (such as dollars). This is indeed the case in Wilson's problem, where all four goals are measured in dollars. However, what happens if different goals are measured in different units? For example, the first goal could be about sales (measured in dollars), and the second goal could be about steel usage (measured in pounds). In such cases, it is very difficult to assign appropriate weights because different deviation variables in the same objective function are measured in different units.

It is not always easy to assign suitable weights for the different deviation variables.

Second, even if all goals are measured in the same units, it is not always easy to assign suitable weights for the different deviation variables. For example, in Wilson's problem, how does management decide that the total sales goal is exactly 5 times as important as the other three goals? What if it is only 2.5 times as important? Clearly, this would affect the choice of weights, which, in turn, could affect the optimal solution.

SCREENSHOT 6-6A **Excel Layout and Solver Entries for Wilson Doors—Weighted Goals Solution 1**



Wilson Doors (Weighted GP #1)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
3		E	I	C	d_T^-	d_T^+	d_E^-	d_E^+	d_I^-	d_I^+	d_C^-	d_C^+				
4		Exterior doors	Interior doors	Comm doors	Under ach total sales	Over ach total sales	Under ach exter doors	Over ach exter doors	Under ach inter doors	Over ach inter doors	Under ach comm doors	Over ach comm doors				
5	Solution value	1000.00	800.00	200.00	0.00	0.00	0.00	0.00	0.00	28000.00	13000.00	0.00				
6	Goal weights				5		1		1		1		13000.00			
7	Constraints:															Achieved
8	Total sales goal	70	110	110	1	-1							180000.00	=	180000	180000.00
9	Exterior doors goal	70					1	-1					70000.00	=	70000	70000.00
10	Interior doors goal		110						1	-1			60000.00	=	60000	88000.00
11	Comm doors goal			110							1	-1	35000.00	=	35000	22000.00
12	Steel usage	4	3	7									7800.00	<=	9000	
13	Forming time	2	4	3									5800.00	<=	6000	
14	Assembly time	2	3	4									5200.00	<=	5200	
15													LHS	Sign	RHS	

File: 6-6.xls, sheet: 6-6B

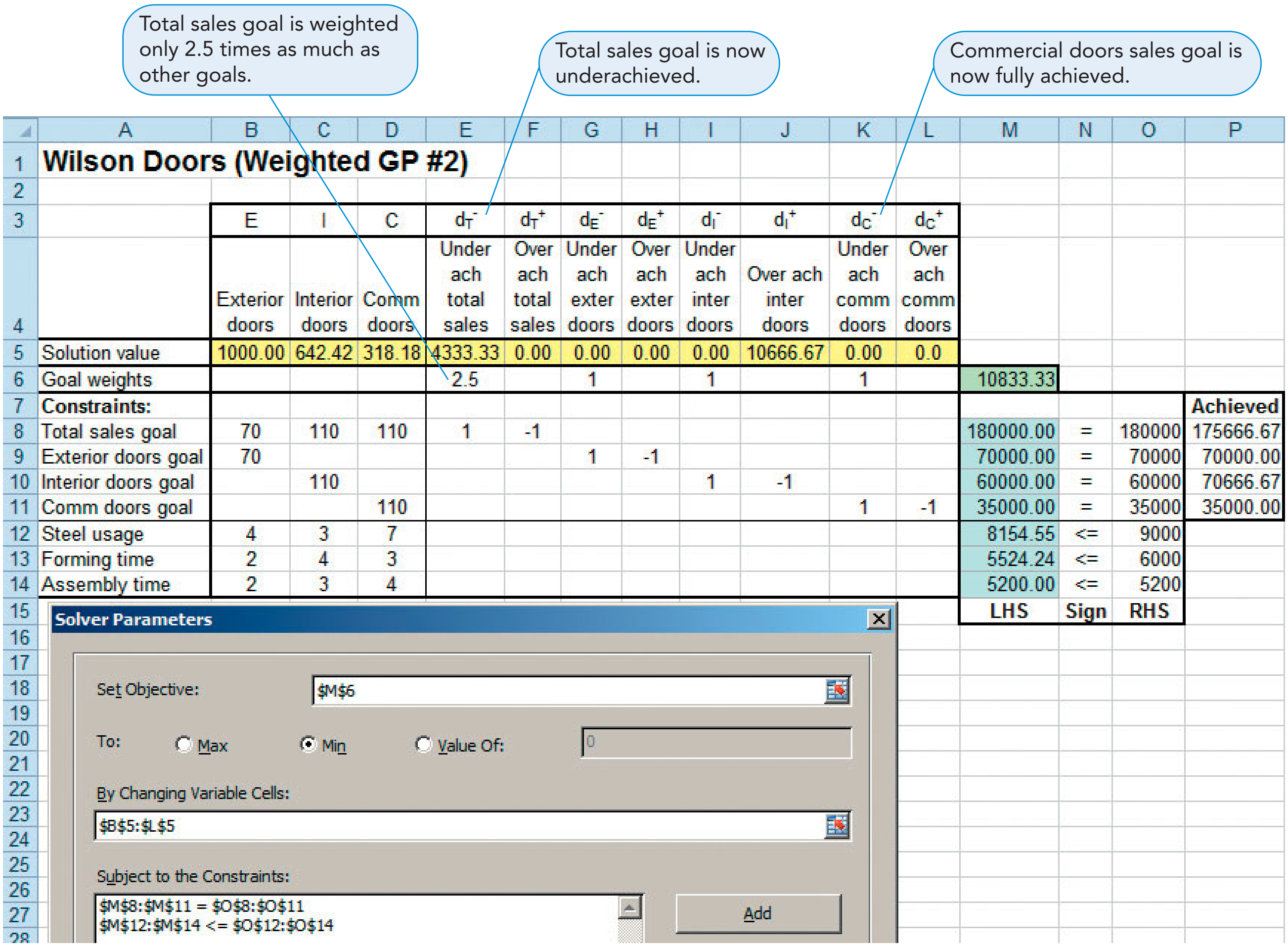
In fact, as shown in Screenshot 6-6B, if we assign a weight of only 2.5 (instead of 5) to the total sales goal in Wilson's weighted GP model and continue to assign a weight of 1 to each of the other three goals, the optimal solution changes completely. Interestingly, the total sales goal, which Wilson has specified as the most important goal, now turns out to be the only goal that is underachieved (by $4,333.33). The exterior and commercial doors sales goals are fully satisfied, while the interior doors sales goal is actually overachieved by $10,666.67. This clearly illustrates the importance of properly selecting weights.

By the way, the LP solution shown in Screenshot 6-6B has fractional solution values for interior and commercial doors. Wilson can fix this either by solving the problem as a general IP model or by rounding off the fractional values appropriately. For your convenience, the IP solution (obtained by constraining variables *E*, *I*, and *C* to be integer valued in Solver) for this problem is included in the Excel file *6-6.xls* on the Companion Website for this textbook; see the worksheet named *6-6B IP*. The total sales goal turns out to be underachieved by $4,290 in the IP solution.

We use ranks when it is difficult to assign weights for deviation variables.

To overcome these two drawbacks with the weighted GP approach, we examine an alternate approach—the ranked, or prioritized, goals approach—for solving GP problems.

SCREENSHOT 6-6B **Excel Layout and Solver Entries for Wilson Doors—Weighted Goals Solution 2**



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	**Wilson Doors (Weighted GP #2)**															
2																
3		E	I	C	d_T^-	d_T^+	d_E^-	d_E^+	d_I^-	d_I^+	d_C^-	d_C^+				
4		Exterior doors	Interior doors	Comm doors	Under ach total sales	Over ach total sales	Under ach exter doors	Over ach exter doors	Under ach inter doors	Over ach inter doors	Under ach comm doors	Over ach comm doors				
5	Solution value	1000.00	642.42	318.18	4333.33	0.00	0.00	0.00	0.00	10666.67	0.00	0.0				
6	Goal weights				2.5		1		1		1		10833.33			
7	**Constraints:**															**Achieved**
8	Total sales goal	70	110	110	1	-1							180000.00	=	180000	175666.67
9	Exterior doors goal	70					1	-1					70000.00	=	70000	70000.00
10	Interior doors goal		110						1	-1			60000.00	=	60000	70666.67
11	Comm doors goal			110							1	-1	35000.00	=	35000	35000.00
12	Steel usage	4	3	7									8154.55	<=	9000	
13	Forming time	2	4	3									5524.24	<=	6000	
14	Assembly time	2	3	4									5200.00	<=	5200	
15													**LHS**	**Sign**	**RHS**	

IN ACTION Goal Programming Helps NBC Increase Revenues and Productivity

In 2000, the National Broadcasting Company's (NBC) television and cable networks, TV stations, and Internet divisions generated more than $5 billion in revenues for its parent company General Electric. Of these, the television network business was by far the largest, contributing more than $4 billion in revenues.

Recognizing the need for a system that would generate sales plans quickly to meet all client requirements and make optimal use of the available inventory, researchers modeled NBC's sales planning problem as a goal program. The system was designed to help management adjust pricing dynamically based on the current market and inventory situations. Client requests were modeled as goal constraints, with penalties associated with missing these goals. Penalties were linearly proportional to the magnitude of deviation from the goals, and varied also based on the importance of the client requirements.

These systems have now become an integral part of sales processes at NBC, and it is estimated that from May 1996 to June 2000, NBC used them to generate sales plans and manage inventory worth more than $9 billion, resulting in net gains of over $200 million.

Source: Based on S. Bollapragada et al. "NBC's Optimization Systems Increase Revenues and Productivity," *Interfaces* 32, 1 (January–February 2002): 47–60.

Solving Goal Programming Models with Ranked Goals

Lower-ranked goals are considered only after higher-ranked goals are met.

In the ranked goals approach to solving GP models, we assign ranks (or priorities), rather than weights, to goals. The idea is that goals can be ranked based on their importance to management. Lower-ranked goals are considered only after higher-ranked goals are met. Note that it is possible to assign the same rank to two or more goals.

Let us discuss this approach by revisiting Wilson Doors Company's problem. Recall that Wilson's management has currently specified the following four goals:

Goal 1: Achieve total sales of at least \$180,000

Goal 2: Achieve exterior doors sales of at least \$70,000

Goal 3: Achieve interior doors sales of at least \$60,000

Goal 4: Achieve commercial doors sales of at least \$35,000

The ranked goals approach can handle goals that are measured in different units.

Because in the ranked goals approach we are no longer restricted to measuring all goals in the same units, let us expand Wilson's problem by adding another goal. Suppose Wilson plans to switch to a different type of steel for the next production period. Management would therefore like to ensure that the production plan this period uses up as much of the current availability of steel (9,000 pounds) as possible. This is formally stated in the following goal:

Goal 5: Achieve steel usage of as close to 9,000 pounds as possible

Wilson's management has examined these five goals and has decided to rank, them in decreasing order of rank, as follows:

Rank R_1: Goal 1

Rank R_2: Goal 5

Rank R_3: Goals 2, 3, and 4

This means, in effect, that meeting the total sales goal is much more important than meeting the steel usage goal which, in turn, is much more important than meeting the sales goals for each of the three types of doors. If we wish, we can further distinguish between goals within the same rank by assigning appropriate weights. For example, we can assign appropriate weights to any of the three goals with rank R_3 (i.e., goals 2, 3, and 4) to make that goal more important than the other two.

FORMULATING THE RANKED GP MODEL In addition to the eight deviation variables that we have already defined earlier (i.e., d_T^-, d_T^+, d_E^-, d_E^+, d_I^-, d_C^+, d_C^-, and d_C^+), we define a ninth deviation variable, as follows:

$$d_S^- = \text{amount by which the steel usage goal is underachieved}$$

Note that we do not have to define a deviation variable for overachievement of steel usage (i.e., d_s^+) because steel is a resource constraint. That is, steel usage can never exceed 9,000 pounds. Also, unlike the eight deviation variables associated with the four sales goals, which are measured in dollars, the deviation variable d_S^- is measured in pounds.

Using the deviation variable d_S^- we can express the steel usage goal mathematically, as follows, just as we expressed the other four goals:

$$4E + 3I + 7C + d_S^- = 9{,}000 \qquad \text{(steel usage goal)}$$

Based on the specified ranking of goals (recall that goals with rank R_1 are the most important, goals with rank R_2 are the next most important, then R_3, and so on), Wilson's ranked GP problem can be stated as

$$\text{Minimize ranked deviations} = R_1(d_T^-) + R_2(d_S^-) + R_3(d_E^- + d_I^- + d_C^-)$$

subject to the constraints

$$\begin{aligned}
70E + 110I + 110C + d_T^- - d_T^+ &= 180{,}000 && \text{(total sales goal)}\\
4E + 3I + 7C + d_S^- &= 9{,}000 && \text{(steel usage goal)}\\
70E + d_E^- - d_E^+ &= 70{,}000 && \text{(exterior doors sales goal)}
\end{aligned}$$

$$
\begin{aligned}
110I + d_I^- - d_I^+ &= 60{,}000 && \text{(interior doors sales goal)}\\
110C + d_C^- - d_C^+ &= 35{,}000 && \text{(commercial doors sales goal)}\\
2E + 4I + 3C &\leq 6{,}000 && \text{(forming time)}\\
2E + 3I + 4C &\leq 5{,}200 && \text{(assembly time)}\\
E, I, C, d_T^-, d_T^+, d_S^-, d_E^-, d_E^+, d_I^-, d_I^+, d_C^-, d_C^+ &\geq 0 && \text{(nonnegativity)}
\end{aligned}
$$

Note that within each rank, the objective function in this model includes only the underachievement deviation variable because all four sales goals specify that the goals should be "at least" met, and the steel usage goal can never be overachieved.

Solving a model with ranked goals requires us to solve a series of LP models.

Rank R_1 goals are considered first.

SOLVING THE RANK R_1 GP MODEL AND INTERPRETING THE RESULTS To find the optimal solution for a GP model with ranked goals, we need to set up and solve a series of LP models. In the first of these LP models, we consider only the highest ranked (rank R) goals and ignore all other goals (ranks R_2 and R_3). The objective function then includes only the deviation variable with rank R_1. In Wilson's problem, the objective of the first LP model is

$$\text{Minimize rank } R_1 \text{ deviation} = d_T^-$$

WWW

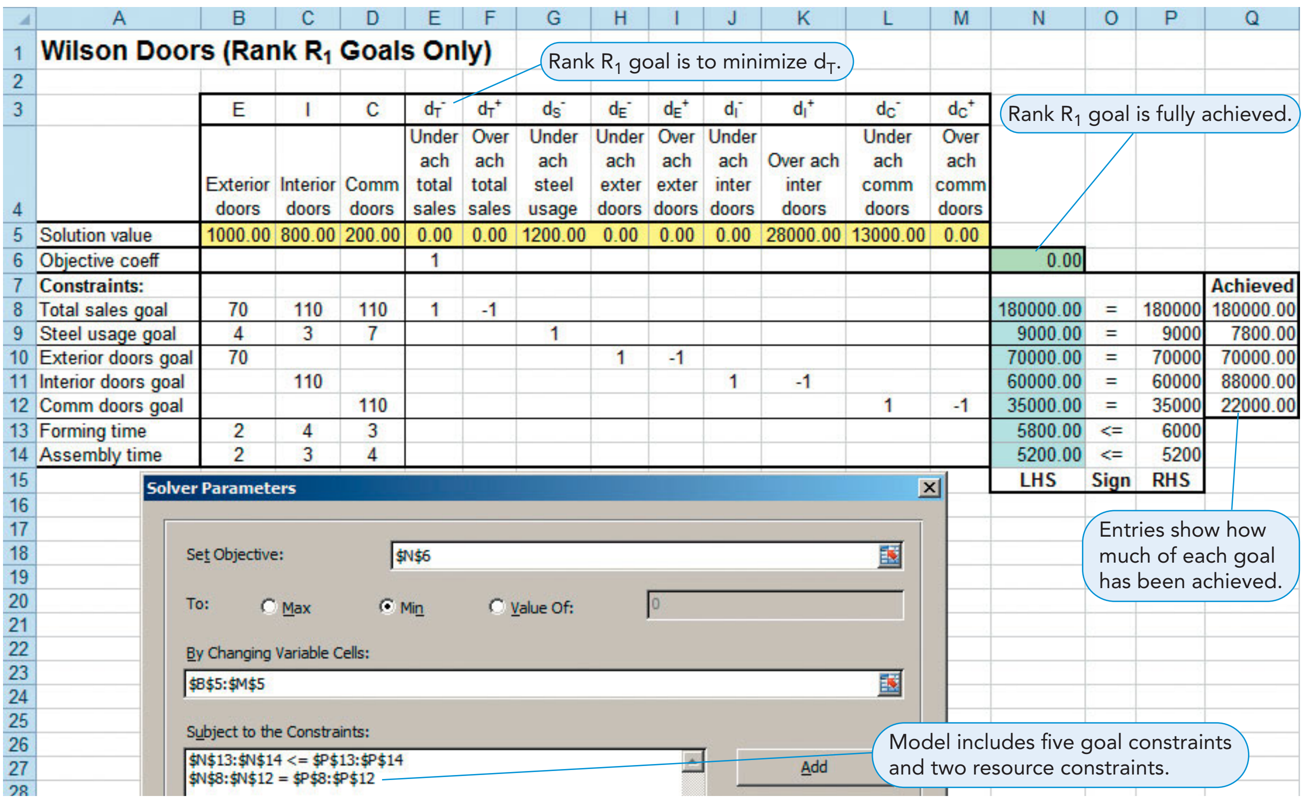
File: 6-7.xls, sheet: 6-7A

Solving this LP model using Solver is a rather simple task, and Screenshot 6-7A shows the relevant information. The results show that it is possible to fully achieve the rank R_1 goal (i.e., the total sales goal can be fully satisfied and the optimal value of d_T^- is 0). However, at the present time, the steel usage goal is underachieved by 1,200 pounds, the interior doors sales goal is overachieved by \$28,000, and the commercial doors sales goal is underachieved by \$13,000.

Rank R_2 goals are considered next. Optimal values of rank R_1 goals are explicitly specified in the model.

SOLVING THE RANK R_2 GP MODEL AND INTERPRETING THE RESULTS Now that we have optimally solved the model with the rank R_1 goal, we consider all goals with the next-highest rank (R_2) in the second LP model. In Wilson's problem, this is the steel usage goal. However, in setting up this LP model, we explicitly specify the optimal value of the total sales goal from

SCREENSHOT 6-7A **Excel Layout and Solver Entries for Wilson Doors—Rank R_1 Goals Only**



Wilson Doors (Rank R1 Goals Only)

	E	I	C	d_T^-	d_T^+	d_S^-	d_E^-	d_E^+	d_I^-	d_I^+	d_C^-	d_C^+				
	Exterior doors	Interior doors	Comm doors	Under ach total sales	Over ach total sales	Under ach steel usage	Under ach exter doors	Over ach exter doors	Under ach inter doors	Over ach inter doors	Under ach comm doors	Over ach comm doors				
Solution value	1000.00	800.00	200.00	0.00	0.00	1200.00	0.00	0.00	0.00	28000.00	13000.00	0.00				
Objective coeff				1									0.00			
Constraints:																Achieved
Total sales goal	70	110	110	1	-1								180000.00	=	180000	180000.00
Steel usage goal	4	3	7			1							9000.00	=	9000	7800.00
Exterior doors goal	70						1	-1					70000.00	=	70000	70000.00
Interior doors goal		110							1	-1			60000.00	=	60000	88000.00
Comm doors goal			110								1	-1	35000.00	=	35000	22000.00
Forming time	2	4	3										5800.00	<=	6000	
Assembly time	2	3	4										5200.00	<=	5200	
													LHS	Sign	RHS	

the rank R_1 model. To do so, we set the value of the relevant deviation variable (i.e., d_T^-) to its optimal value of 0 in the LP model.

For Wilson's second LP model, the objective function and *additional* constraint are as follows:

$$\text{Minimize rank } R_2 \text{ deviation } = d_S^-$$

and

$$d_T^- = 0 \qquad (\text{optimal value of rank } R_1 \text{ goal})$$

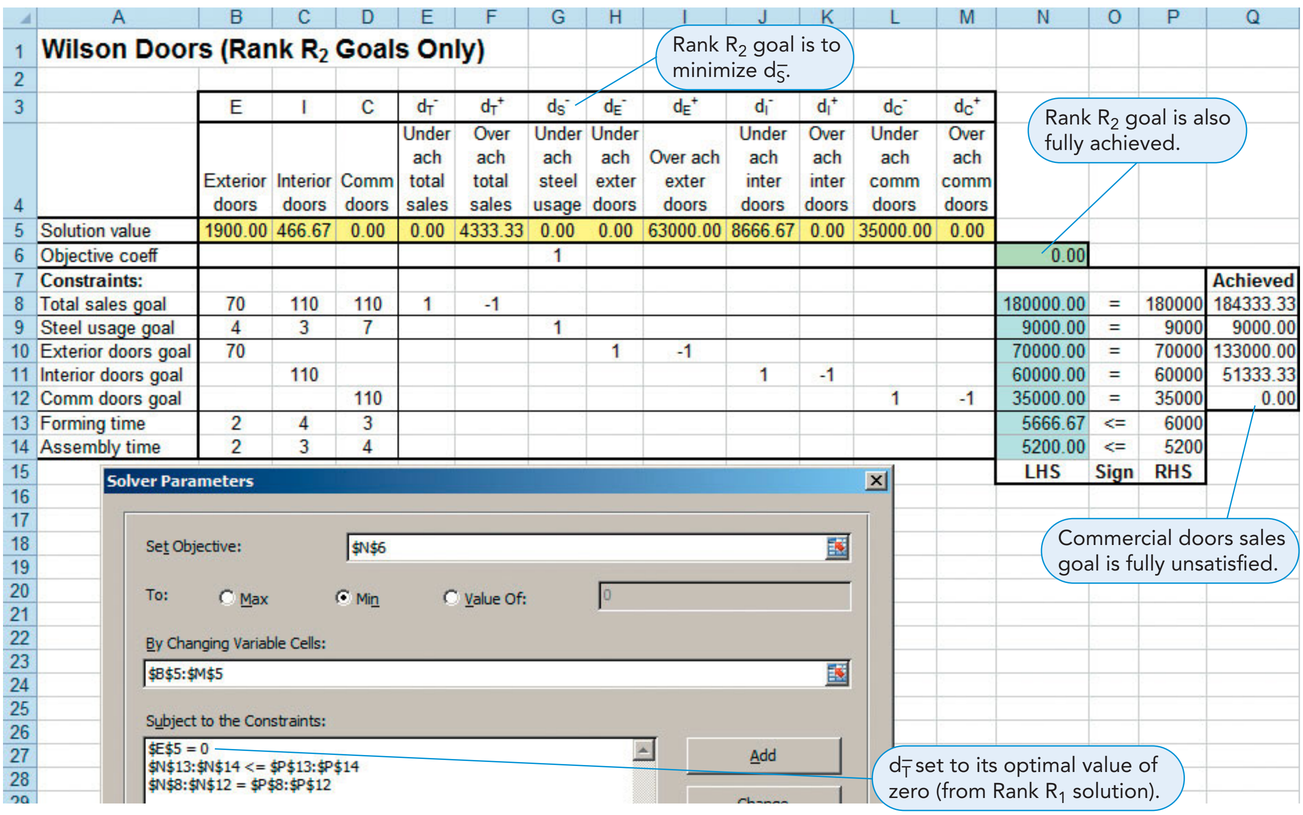
File: 6-7.xls, sheet: 6-7B

Screenshot 6-7B shows the Excel layout and Solver entries for this LP model. The results show that it is possible to fully achieve the rank R_2 goal also. That is, it is possible to reduce the value of the deviation variable d_S^- also to 0, while maintaining the value of the rank R_1 deviation variable d_T^- at its optimal value of 0. In fact, the total sales goal is now overachieved by \$4,333.33, and the exterior doors sales goal is overachieved by \$63,000.

However, this emphasis on reducing the value of d_S^- results in the value of d_C^- ballooning up from \$13,000 in the rank R_1 solution (Screenshot 6-7A) to \$35,000 in the rank R_2 solution (Screenshot 6-7B). This implies that the commercial doors sales goal is fully unsatisfied and that no commercial doors should be made. Likewise, the interior doors sales goal is also underachieved by \$8,666.67. While this solution may seem unfair to the sales agents for interior and commercial doors, it is still perfectly logical because Wilson has ranked the steel usage goal higher than the sales goals for all three door types.

The LP solution shown in Screenshot 6-7B has a fractional solution value for interior doors. Interestingly, the IP solution for this problem (which is included in the Excel file *6-7.xls* on the Companion Website for this textbook; see the worksheet named *6-7B IP*) is considerably different from the LP solution in Screenshot 6-7B. The steel usage goal is still fully satisfied. However, while overachievements d_T^+ and $d_E^+ d$ and underachievement d_C^- all decrease from their corresponding LP solution values, underachievement d_I^- increases from \$8,666.67 to \$13,580.

SCREENSHOT 6-7B **Excel Layout and Solver Entries for Wilson Door—Rank R_2 Goals Only**



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	Wilson Doors (Rank R_2 Goals Only)																
2																	
3		E	I	C	d_T^-	d_T^+	d_S^-	d_E^-	d_E^+	d_I^-	d_I^+	d_C^-	d_C^+				
4		Exterior doors	Interior doors	Comm doors	Under ach total sales	Over ach total sales	Under ach steel usage	Under ach exter doors	Over ach exter doors	Under ach inter doors	Over ach inter doors	Under ach comm doors	Over ach comm doors				
5	Solution value	1900.00	466.67	0.00	0.00	4333.33	0.00	0.00	63000.00	8666.67	0.00	35000.00	0.00				
6	Objective coeff						1							0.00			
7	Constraints:																Achieved
8	Total sales goal	70	110	110	1	-1								180000.00	=	180000	184333.33
9	Steel usage goal	4	3	7			1							9000.00	=	9000	9000.00
10	Exterior doors goal	70						1	-1					70000.00	=	70000	133000.00
11	Interior doors goal		110							1	-1			60000.00	=	60000	51333.33
12	Comm doors goal			110								1	-1	35000.00	=	35000	0.00
13	Forming time	2	4	3										5666.67	<=	6000	
14	Assembly time	2	3	4										5200.00	<=	5200	
15														LHS	Sign	RHS	

That is, the IP solution pulls three of the four goals closer to their target while moving one further away from its target, when compared to the LP solution.

R_3 goals are now considered. Optimal values of rank R_1 and R_2 goals are explicitly specified in the model.

SOLVING THE RANK R_3 GP MODEL AND INTERPRETING THE RESULTS Now that the goals with ranks R_1 and R_2 have been optimized, we now consider all goals with the next-highest rank (R_3) in the third LP model. As before, in setting up this model, we explicitly specify the optimal values of the rank R_1 and R_2 goals obtained from the first two LP models.

For Wilson's third LP model, the objective function and *additional* constraints are as follows:

$$\text{Minimize rank } R_3 \text{ deviation} = d_E^- + d_I^- + d_C^-$$

and

$$d_T^- = 0 \qquad \text{(optimal value of rank } R_1 \text{ goal)}$$
$$d_S^- = 0 \qquad \text{(optimal value of rank } R_2 \text{ goal)}$$

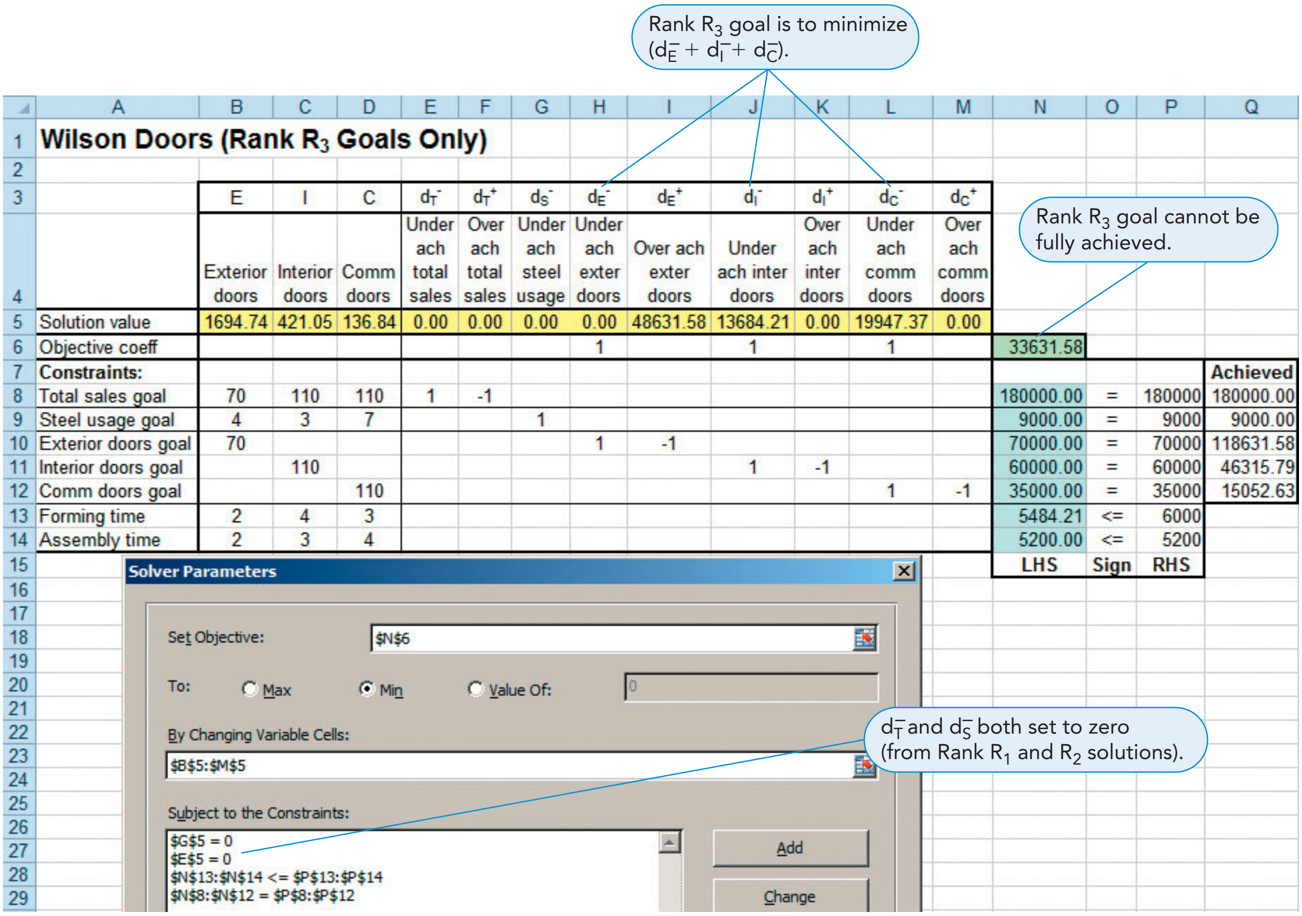
WWW

File: 6-7.xls, sheet: 6-7C

Screenshot 6-7C shows the Excel layout and Solver entries for this LP model. The results show that after fully optimizing the rank R_1 and R_2 goals, the best we can do is to achieve a total underachievement of \$33,631.58 in the rank R_3 goals. In the final solution, the total sales goal is exactly satisfied while the exterior doors sales goal is overachieved by \$48,631.58. In contrast, the interior doors and commercial doors sales goals are underachieved by \$13,684.21 and \$19,947.37, respectively.

As with the second LP model, this solution too has fractional values for the production variables. The IP solution for this problem (which is included in the Excel file *6-7.xls* on the Companion Website for this textbook; see the worksheet named *6-7C IP*) turns out to be the

SCREENSHOT 6-7C Excel Layout and Solver Entries for Wilson Doors—Rank R_3 Goals Only



Wilson Doors (Rank R_3 Goals Only)

	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
	E	I	C	d_T^-	d_T^+	d_S^-	d_E^-	d_E^+	d_I^-	d_I^+	d_C^-	d_C^+				
	Exterior doors	Interior doors	Comm doors	Under ach total sales	Over ach total sales	Under ach steel usage	Under ach exter doors	Over ach exter doors	Under ach inter doors	Over ach inter doors	Under ach comm doors	Over ach comm doors				
Solution value	1694.74	421.05	136.84	0.00	0.00	0.00	0.00	48631.58	13684.21	0.00	19947.37	0.00				
Objective coeff							1		1		1		33631.58			
Constraints:																**Achieved**
Total sales goal	70	110	110	1	-1								180000.00	=	180000	180000.00
Steel usage goal	4	3	7			1							9000.00	=	9000	9000.00
Exterior doors goal	70						1	-1					70000.00	=	70000	118631.58
Interior doors goal		110							1	-1			60000.00	=	60000	46315.79
Comm doors goal			110								1	-1	35000.00	=	35000	15052.63
Forming time	2	4	3										5484.21	<=	6000	
Assembly time	2	3	4										5200.00	<=	5200	
													LHS	**Sign**	**RHS**	

same as the IP solution we obtained for the rank R_2 model. That is, if we solve Wilson's problem as IP models, the rank R_3 model is not able to improve on the solution obtained in the rank R_2 model.

Comparing the Two Approaches for Solving GP Models

The weighted goals approach considers all goals simultaneously, and the optimal solution depends to a great extent on the weights assigned for different goals. In contrast, the ranked goals approach considers goals in a hierarchical manner. Optimal values for all higher-ranked goal deviation variables are *explicitly* specified while considering LP models with lower-ranked goals as objective functions. Which approach should we then use for a specific problem? If all goals are measured in the same units, and if it is possible to assign appropriate weights for each goal, using the weighted goals approach is clearly the easier option. In all other situations, we would need to use the ranked goals approach.

6.6 Nonlinear Programming Models

In many real-world problems, the objective function and/or one or more constraints may be nonlinear.

LP, IP, and GP all assume that a problem's objective function and constraints are linear. That means that they cannot contain nonlinear terms, such as X^3, $1/X$, $\log X$, or $5XY$. Yet in many real-world situations, the objective function and/or one or more constraints may be nonlinear. Here are two simple examples:

- We have assumed in all models so far that the profit contribution per unit of a product is fixed, regardless of how many units we make of the product. That is, if Y denotes the number of units made of a specific product and the product has a profit contribution of \$6 per unit, the total profit is \$6Y, for *all* values of Y. However, it is likely that the unit profit contribution of a product decreases as its supply (i.e., number of units made) increases. Suppose this relationship turns out to be

$$\text{Profit contribution per unit} = \$6 - \$0.02Y$$

Then, the total profit from this product is given by the following nonlinear expression:

$$\text{Total profit} = (\$6 - \$0.02Y) \times Y = \$6Y - \$0.02Y^2$$

- Likewise, we have assumed in all models so far that the relationship between resource usage and production level is linear. For example, if each patient requires 5 minutes of nursing time and there are P patients, the total time needed is $5P$ minutes, for all values of P. This term would be included in the LHS of the nursing time constraint. However, it is quite possible that the efficiency of nurses decreases as the patient load increases. Suppose the time required per patient is actually $(5 + 0.25P)$. That is, the time per patient increases as the number of patients increases. The term to be included in the nursing time constraint's LHS would now be $(5 + 0.25P) \times P = (5P + 0.25P^2)$, which would make the constraint nonlinear.

In such situations, the resulting model is called a **nonlinear programming (NLP)** model. By definition, an NLP model has a nonlinear objective function, or at least one nonlinear constraint, or both. In this section, we examine NLP models and also illustrate how Excel's Solver can often be used to solve these models. In practice, NLP models are difficult to solve and should be used with a lot of caution. Let us first examine the reason for this difficulty.

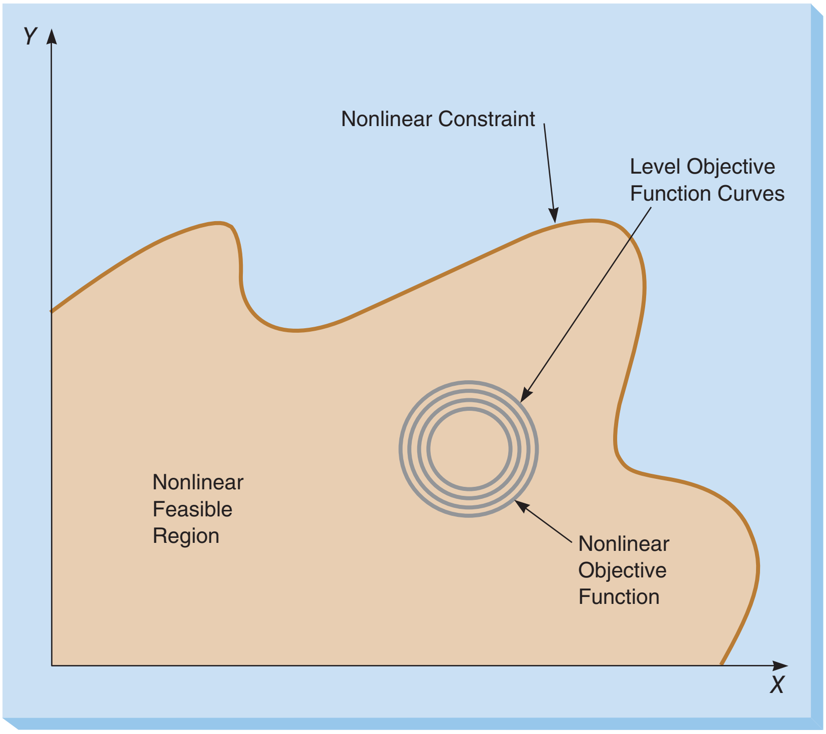
Why Are NLP Models Difficult to Solve?

In every LP, IP, and GP model, the objective function and all constraints are linear. This implies, for example, that with two variables, each equation in the model corresponds to a straight line. In contrast, as shown in Figure 6.2, a nonlinear expression in two variables is a curve. Depending on the extent of nonlinearity in the expression, the curve could be quite pronounced in that it could have many twists and turns.

The optimal solution to an NLP model need not be at a corner point of the feasible region.

You may recall from Chapter 2 that a feature of all LP models is that an optimal solution always occurs at a corner point (i.e., point where two or more linear constraints intersect).

FIGURE 6.2
Model with Nonlinear Constraints and a Nonlinear Objective Function



Software packages (including Solver) exploit this feature to find optimal solutions quickly even for large linear models. Unfortunately, if one or more constraints are nonlinear, an optimal solution need not be at a corner point of the feasible region. Further, as you can see from Figure 6.2, if the objective function itself is nonlinear (as in the equation of an ellipse or a sphere), it is not even easy to visualize at which feasible point the solution is optimized. This is one major reason why many NLP models are so difficult to solve in practice. As you can well imagine, this issue becomes even more difficult for NLP models that involve more decision variables.

LOCAL VERSUS GLOBAL OPTIMAL SOLUTIONS A second reason for the difficulty in solving NLP models is the concept of local versus global optimal solutions. Perhaps a simple analogy will help you understand this concept. A local optimal solution is like the peak of a specific mountain in a mountain range. The global optimal solution, in contrast, is the peak of the highest mountain in that range. If you are on a specific mountain, it is likely that you can easily see the peak of that mountain—and possibly even find your way to it. However, unless you are able to see all the mountains in the entire range from your current location, you have no way of knowing if the peak of your specific mountain is just a local peak or whether it is the global peak.

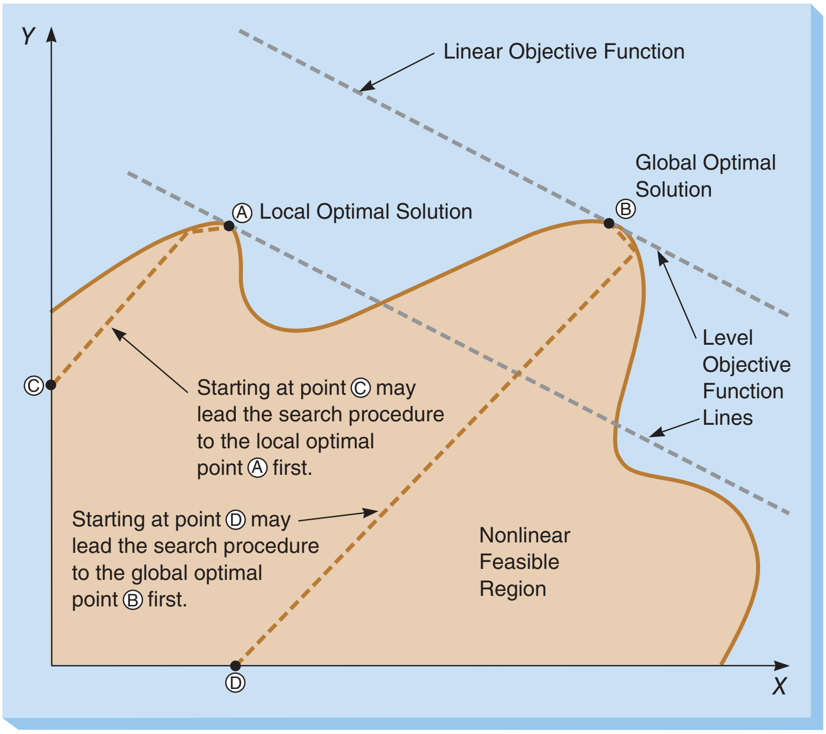
An NLP model can have both local and global optimal solutions.

Figure 6.3 illustrates this phenomenon with respect to NLP models. For the linear objective function shown in the figure, point Ⓐ is a local optimal solution, whereas point Ⓑ is a global optimal solution. The difficulty with all NLP solution procedures (including the procedure available in Solver) is that depending on where the procedure starts the search process, it could terminate the search at either a global or a local optimal solution. For example, if the procedure starts at point Ⓓ, the search process could in fact lead it to the global optimal solution, point Ⓑ, first. In contrast, if it starts at point Ⓒ, the search process could find the local optimal solution, point Ⓐ, first. Because there are no better solutions in the immediate vicinity of point Ⓐ, the procedure will erroneously terminate and yield point Ⓐ as the optimal solution.

There is no precise way to know where to start the solution search process for an NLP model.

Unfortunately, there is no precise way of knowing where to start the search process for a given NLP problem. Hence, it is usually a good idea to try different starting solutions for NLP models. Hopefully, at least one of them will result in the global optimal solution.

FIGURE 6.3
Local versus Global Optimal Solutions in an NLP Model



Solving Nonlinear Programming Models Using Solver

To illustrate how NLP models can be set up and solved using Solver, let us consider an example in which the objective function and some of the constraints are nonlinear. The weekly profit at Pickens Memorial Hospital depends on the number of patients admitted in three separate categories: medical, surgical, and pediatric. The hospital can admit a total of 200 patients (regardless of category) each week. However, because Pickens Memorial serves a large community, patient demand in each category by itself far exceeds the total patient capacity.

Due to a fixed overhead, the profit per patient in each category actually increases as the number of patients increases. Further, some patients who are initially classified as medical patients then get reclassified as surgical patients. As a result, the profit per surgical patient also

IN ACTION Using Quadratic Programming to Improve Water-Release Policies on the Delaware River

The Delaware River provides half of the drinking water for New York City (NYC). The water releases from three NYC dams on the river's headwaters impact the reliability of the water supply, the flood potential, and the quality of the aquatic habitat. Changes in release policies are, however, restricted due to two US Supreme Court decrees and the need for unanimity among NYC as well as the four states (New York, New Jersey, Pennsylvania, and Delaware) affected by these changes.

In January 2006, a coalition of four conservation organizations undertook a decision modeling–based project to study and suggest revisions to the release policies. A key component of this analysis was a quadratic nonlinear programming allocation model. The primary objective was to benefit river habitat and fisheries without increasing NYC's drought risk. The strategy was to quantify the risk-benefit trade-offs from increased conservation releases and create a simple algorithm that would explicitly link release quantities to reservoir levels.

It is estimated that the use of this model has increased critical summertime fish habitats by about 200 percent, while increasing NYC's drought risk by only 3 percent. The new release rules also mitigate flood risk and are significantly simpler to administer than prior approaches.

Source: Based on P. Kolesar and J. Serio. "Breaking the Deadlock: Improving Water-Release Policies on the Delaware River Through Operations Research," *Interfaces* 41, 1 (January–February 2011): 18–34.

depends on the number of medical patients admitted. The accountants at Pickens Memorial have analyzed this situation and have identified the following information:

$$\begin{aligned}\text{Profit contribution per medical patient} &= \$45 + \$2M\\ \text{Profit contribution per surgical patient} &= \$70 + \$3S + \$2M\\ \text{Profit contribution per pediatric patient} &= \$60 + \$3P\end{aligned}$$

where

$$\begin{aligned}M &= \text{number of medical patients admitted}\\ S &= \text{number of surgical patients admitted}\\ P &= \text{number of pediatric patients admitted}\end{aligned}$$

Pickens Memorial has identified three main constraints for this model: x-ray capacity, marketing budget, and lab capacity. Table 6.8 shows the relevant weekly data for these three constraints for each category of patient. The table also shows the weekly availabilities of each of these three resources.

The hospital's chief laboratory supervisor has noted that the time required per lab test increases as the total number of medical patients admitted (M) increases. Based on historical data, the supervisor estimates this relationship to be as follows

$$\text{Time required per lab test (in hours)} = 0.2 + 0.001M$$

In this NLP example, the objective function as well as some of the constraints are nonlinear.

FORMULATING THE PROBLEM The objective function for Pickens Memorial seeks to maximize the total profit and can be written as

$$\begin{aligned}\text{Maximize profit} &= (\$45 + \$2M) \times M + (\$70 + \$3S + \$2M) \times S + (\$60 + \$3P) \times P\\ &= \$45M + \$2M^2 + \$70S + \$3S^2 + \$2MS + \$60P + \$3P^2\end{aligned}$$

Clearly, this is a nonlinear expression. The constraints correspond to the total patient capacity of 200 and to the three limiting resources (i.e., x-ray capacity, marketing budget, and lab capacity). They may be expressed as follows:

$$\begin{aligned}M + S + P &\le 200 && \text{(total patient capacity)}\\ M + 3S + P &\le 560 && \text{(x-ray capacity)}\\ 3M + 5S + 3.5P &\le 1{,}000 && \text{(marketing budger, \$)}\\ (0.2 + 0.001M) \times (3M + 3S + 3P) &\le 140 && \text{(lab capacity, hours)}\\ M, S, P &\ge 0\end{aligned}$$

The total patient capacity, x-ray capacity, and marketing budget constraints are linear. However, the lab capacity constraint is nonlinear because it includes terms involving multiplication of variables. We can simplify and rewrite this constraint as

$$0.6M + 0.6S + 0.6P + 0.003M^2 + 0.003MS + 0.003MP \le 140 \text{ (lab capacity, hours)}$$

File: 6-8.xls, sheet: 6-8A

Solver uses the GRG procedure *to solve NLP models.*

SOLVING THE PROBLEM USING EXCEL'S SOLVER We now illustrate how Excel's Solver can be used to solve this NLP model. Solver uses the **generalized reduced gradient (GRG) procedure**, sometimes called the *steepest ascent* (or *steepest descent*) *procedure*. This is an iterative procedure that moves from one feasible solution to the next in improving the value of the objective function. The GRG procedure can handle problems with both nonlinear constraints and nonlinear objective functions.

TABLE 6.8
Data for Pickens Memorial Hospital

	MEDICAL	SURGICAL	PEDIATRIC	AVAILABILITY
Number of x-rays per patient	1	3	1	560 x-rays
Marketing budget per patient	\$3	\$5	\$3.5	\$1,000
Number of lab tests per patient	3	3	3	140 hours

The Excel layout includes several nonlinear terms involving the decision variables.

There are only three decision variables (i.e., M, S, and P) in Pickens Memorial's NLP model. These are denoted by cells B5, C5, and D5, respectively, in Screenshot 6-8A. However, the model includes several nonlinear terms involving these three variables: M^2, S^2, P^2, MS, and MP. There are several ways in which we can include these terms in our Excel layout. Here are two simple approaches (*Note:* Screenshot 6-8A illustrates the second approach):

- We can directly type the nonlinear formula in the appropriate cell. For example, for the nonlinear objective function in this model, the formula in the objective cell can be directly entered as follows:

$$= 45*B5 + 2*B5\text{^}2 + 70*C5 + 3*C5\text{^}2 + 2*B5*C5 + 60*D5 + 3*D5\text{^}2$$

  In a similar manner, we can enter the nonlinear formula for the lab capacity constraint directly in the cell corresponding to the LHS of that constraint, as follows:

$$\begin{aligned} &= 0.6*B5 + 0.6*C5 + 0.6*D5 + 0.003*B5\text{^}2 \\ &\quad + 0.003*B5*C5 + 0.003*B5*D5 \end{aligned}$$

The SUMPRODUCT function is again used to calculate all constraint LHS values and the objective function value.

- Alternatively, as illustrated in Screenshot 6-8A, we can use the same Excel layout that we have used in all LP and IP models so far. This means that (1) each decision variable is modeled in a separate column of the worksheet and (2) the objective function and LHS formulas for all constraints are computed using Excel's SUMPRODUCT function. To use this layout for NLP models, we create a cell entry for each linear or nonlinear term involving the decision variables. In Pickens Memorial's case, we need cells for M, S, P, M^2, S^2, P^2, MS, and MP. These terms are represented by cells B8 to I8, respectively, in Screenshot 6-8A. (For clarity, we have shaded these cells blue in this NLP model.) The formulas for these cells are

$$\begin{aligned}
&= B5 && (\text{entry for } M \text{ in cell B8}) \\
&= C5 && (\text{entry for S in cell C8}) \\
&= D5 && (\text{entry for P in cell D8}) \\
&= B5\text{^}2 && (\text{entry for } M^2 \text{ in cell E8}) \\
&= C5\text{^}2 && (\text{entry for } S^2 \text{ in cell F8}) \\
&= D5\text{^}2 && (\text{entry for } P^2 \text{ in cell G8}) \\
&= B5*C5 && (\text{entry for } MS \text{ in cell H8}) \\
&= B5*D5 && (\text{entry for } MP \text{ in cell I8})
\end{aligned}$$

SCREENSHOT 6-8A Formula View of Excel Layout for Pickens Memorial—NLP

These are the only three decision variables in the model.

Entries in columns E to I are nonlinear terms.

Entries in this row are functions of the decision variables.

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Pickens Memorial Hospital (NLP)**											
2												
3		M	S	P								
4		Medical	Surgical	Pediatric								
5	Number of patients											
6												
7		M	S	P	M^2	S^2	P^2	MS	MP			
8	Variable terms	=B5	=C5	=D5	=B5^2	=C5^2	=D5^2	=B5*C5	=B5*D5			
9	Profit	45	70	60	2	3	3	2		=SUMPRODUCT(B9:I9,$B$8:$I$8)		
10	**Constraints:**											
11	Total patients	1	1	1						=SUMPRODUCT(B11:I11,$B$8:$I$8)	<=	200
12	X-ray capacity	1	3	1						=SUMPRODUCT(B12:I12,$B$8:$I$8)	<=	560
13	Marketing budget	3	5	3.5						=SUMPRODUCT(B13:I13,$B$8:$I$8)	<=	1000
14	Lab hours	0.6	0.6	0.6	0.003			0.003	0.003	=SUMPRODUCT(B14:I14,$B$8:$I$8)	<=	140
15										LHS	Sign	RHS

All entries in column J are computed using the SUMPRODUCT function.

GRG Nonlinear must be selected as the solving method in Solver.

The layout for this model now looks similar to all other Excel layouts we have used so far. Hence, we can use the SUMPRODUCT function to model the objective function as well as the constraint LHS values. Note, however, that even though cells B8:I8 are used in computing the objective function and constraint LHS values in column J, only cells B5:D5 are specified in the By Changing Variable Cells box in Solver (as shown in Screenshot 6-8B). The entries in row 8 are simply calculated from the final values for M, S, and P in cells B5:D5, respectively.

File: 6-8.xls, sheet: 6-8B

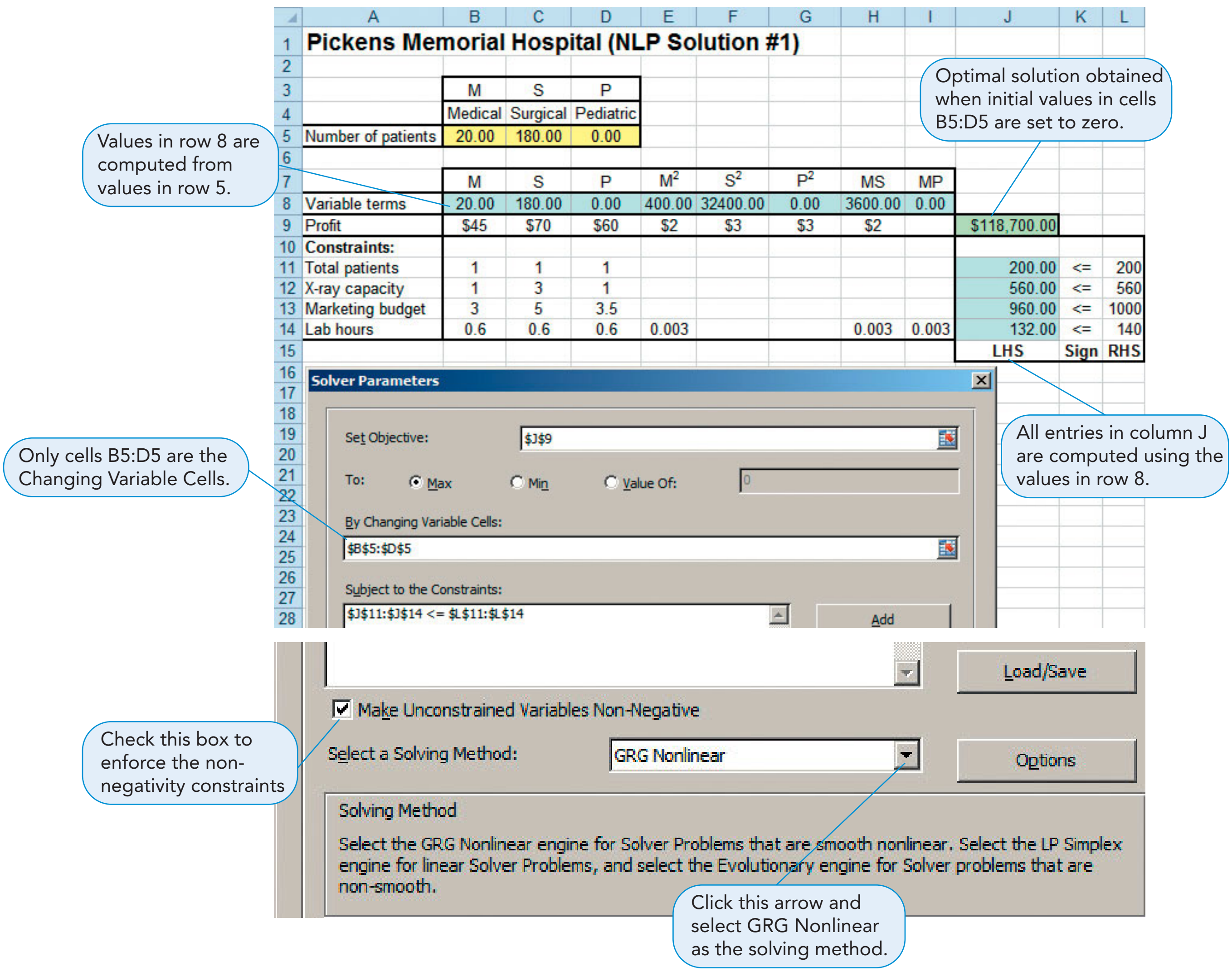
Screenshot 6-8B also shows the other Solver entries and solution for Pickens Memorial's NLP model. We note that in addition to ensuring that the Make Unconstrained Variables Non-Negative box is checked, we must now specify GRG Nonlinear in the Select a Solving Method, instead of Simplex LP, as we have done so far for all LP, IP, and GP models.

INTERPRETING THE RESULTS In obtaining the solution shown in Screenshot 6-8B, we set the initial values of all three decision variables (i.e., cells B5:D5) to zero. The final result indicates that Pickens Memorial should admit 20 medical patients, 180 surgical patients, and no pediatric patients each week, for a total weekly profit of $118,700.

It is usually a good idea to try different starting solutions for NLP models.

Is this a local optimal solution or a global optimal solution? As noted earlier, it is usually a good idea to try different starting solutions for NLP models. Hence, let us solve Pickens Memorial's NLP model again using Solver, but with different starting values for the decision variables. Screenshot 6-8C shows the final result obtained by Solver when we start with initial

SCREENSHOT 6-8B Excel Layout and Solver Entries for Pickens Memorial—NLP Solution 1



SCREENSHOT 6-8C **Excel Layout and Solver Entries for Pickens Memorial—NLP Solution 2**

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Pickens Memorial Hospital (NLP Solution #2)**											
2												
3		M	S	P								
4		Medical	Surgical	Pediatric								
5	Number of patients	0.00	0.00	200.00								
6												
7		M	S	P	M^2	S^2	P^2	MS	MP			
8	Variable terms	0.00	0.00	200.00	0.00	0.00	40000.00	0.00	0.00			
9	Profit	$45	$70	$60	$2	$3	$3	$2		$132,000.00		
10	**Constraints:**											
11	Total patients	1	1	1						200.00	<=	200
12	X-ray capacity	1	3	1						200.00	<=	560
13	Marketing budget	3	5	3.5						700.00	<=	1000
14	Lab hours	0.6	0.6	0.6	0.003			0.003	0.003	120.00	<=	140
15										**LHS**	**Sign**	**RHS**

Solution now recommends admitting only pediatric patients.

Optimal solution obtained when initial values in cells B5:D5 are set to M = 100, S = 0, and P = 100.

File: 6-8.xls, sheet: 6-8C

values of $M = 100$, $S = 0$, and $P = 100$. Interestingly, we get a different final solution now: Pickens Memorial should admit no medical and surgical patients but admit 200 pediatric patients each week, for a total weekly profit of $132,000.

Because this profit is higher than the $118,700 profit shown in Screenshot 6-8B, it is clear that the earlier solution is only a local optimal solution. We can of course manually experiment with other starting values for the decision variables to see if we can get a solution better than $132,000. Solver, however, has an option to automatically try different starting points. We should note, though, that while this approach will identify the best solution from a range of possible local optimal solutions, it does not guarantee that it will find the global optimal solution. In fact, as we will see later, when we use this approach, Solver will explicitly include the message "Solver converged in probability to a global solution" rather than say it found a global optimal solution (as it does when we use the Simplex LP method).

We can use the Multistart option in Solver to automatically try different starting values for the decision variables in an NLP problem.

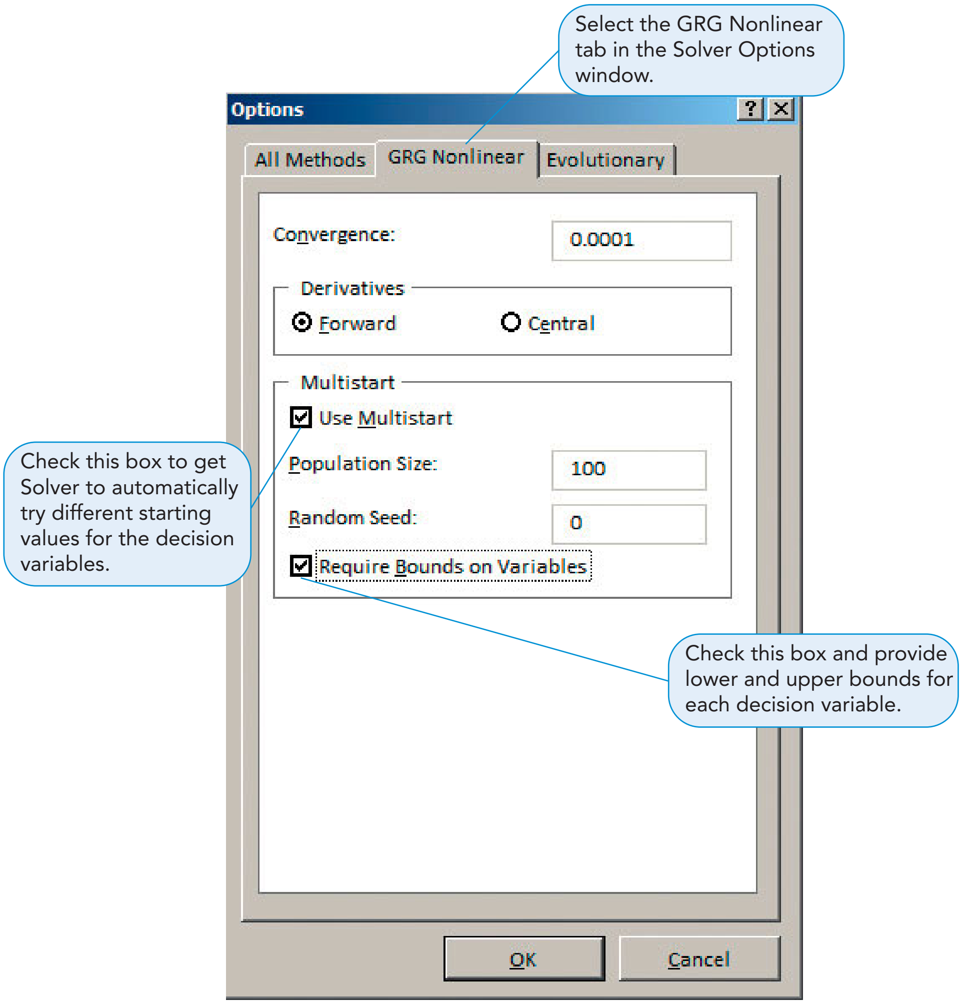
SOLVER OPTIONS FOR NLP MODELS We click the Options button in Solver and select the GRG Nonlinear tab to get the window shown in Screenshot 6-8D. We focus our attention primarily on the box labeled Multistart. To get Solver to automatically try different starting values for the decision variables, we check not only the box labeled Use Multistart but also the box labeled Require Bounds on Variables. What are the bounds on the decision variables? Clearly, the nonnegativity constraints provide a lower bound for each variable. We can easily specify upper bounds for each decision variable by adding them as constraints in Solver. The likelihood of finding a global solution increases as the bounds we specify on variables become tighter, and the longer Solver runs. We illustrate the use of bounds on variables in Solved Problem 6-3 at the end of this chapter.

In the case of Pickens Memorial, it is clear that none of the variables can exceed a value of 200, and we can therefore specify this as the upper bound for all variables. If we do so and solve the model with the options specified as shown in Screenshot 6-8D, we get the same solution shown in Screenshot 6-8C. (We urge you to try this out yourself, using the Excel file *6-8.xls* on the Companion Website for this textbook.) It is therefore likely that this solution is, in fact, the global optimal solution. If so, perhaps Pickens Memorial should consider renaming itself the Pickens Memorial Pediatric Hospital!

Quadratic programming contains squared terms in the objective function.

QUADRATIC PROGRAMMING MODELS When the only nonlinear terms in an objective function are squared terms (such as M^2) and the problem's constraints are all linear, this is a special type of NLP model called a **quadratic programming (QP)** model. A number of useful problems in the field of portfolio selection fall into this category. QP problems can be solved by using a modified version of the simplex method. Such work, however, is beyond the scope of this textbook.

SCREENSHOT 6-8D
Solver Options Window for NLP Models



Computational Procedures for Nonlinear Programming Problems

We cannot always find an optimal solution to an NLP problem.

Although we have used Solver to find an optimal solution for Pickens Memorial's NLP example, there is no general method that guarantees an optimal solution for all NLP problems in a finite number of steps. As noted earlier, NLP problems are inherently more difficult to solve than LP problems.

Perhaps the best way to deal with nonlinear problems is to try to reduce them into a form that is linear or almost linear. One such approach, called *separable programming*, deals with a class of problems in which the objective and constraints are approximated by linear functions. In this way, the powerful procedures (such as the simplex algorithm) for solving LP problems can again be applied. In general, however, work in the area of NLP is the least charted and most difficult of all the decision models.

Summary

This chapter addresses three special types of LP problems. The first, integer programming, examines LP problems that cannot have fractional answers. We note that there are two types of integer variables: general integer variables, which can take on any nonnegative integer value that satisfies all the constraints in a model, and binary variables, which can only take on either of two values: 0 or 1. We illustrate how models involving both types of integer variables can be set up in Excel and solved using Solver.

The second special type of LP problem studied is goal programming. This extension of LP allows problems to have multiple objective functions, each with its own goal. We show how to model such problems using weighted goals as well as ranked goals. In either case, we use Excel's Solver to obtain optimal solutions.

Finally, we introduce the advanced topic of NLP as a special mathematical programming problem. Excel's Solver can be a useful tool in solving simple NLP models.

Glossary

Binary Variables Decision variables that are required to have integer values of either 0 or 1. Also called *0–1 variables*.

Branch-and-Bound (B&B) Method An algorithm used by Solver and other software to solve IP problems. It divides the set of feasible solutions into subregions that are examined systematically.

Deviation Variables Terms that are minimized in a goal programming problem. They are the only terms in the objective function.

Fixed-Charge Problem A problem in which there is a fixed cost in addition to variable costs. Fixed costs need to be modeled using binary (or 0–1) variables.

General Integer Variables Decision variables that are required to be integer valued. Actual values of these variables are restricted only by the constraints in the problem.

Generalized Reduced Gradient (GRG) Procedure A procedure used by Solver to solve NLP problems.

Goal Programming (GP) A mathematical programming technique that permits decision makers to set and rank multiple objective functions.

Integer Programming (IP) A mathematical programming technique that produces integer solutions to LP problems.

Mixed Integer Programming A category of problems in which some decision variables must have integer values (either general integer or binary) and other decision variables can have fractional values.

Nonlinear Programming (NLP) A category of mathematical programming techniques that allow the objective function and/or constraints to be nonlinear.

Quadratic Programming (QP) An NLP model in which the objective function includes only quadratic nonlinear terms and the constraints are all linear.

Ranked Goals An approach in which a decision maker ranks goals based on their relative importance to the decision maker. Lower-ranked goals are considered only after higher-ranked goals have been optimized. Also known as *prioritized goals*.

Satisfice To come as close as possible to reaching a set of objectives.

Weighted Goals An approach in which the decision maker assigns weights to deviation variables based on their relative importance to the decision maker.

Solved Problems

Solved Problem 6-1

Consider the 0–1 integer programming problem that follows:

$$\text{Maximize profit} = 50X_1 + 45X_2 + 48X_3$$

subject to the constraints

$$\begin{aligned} 19X_1 + 27X_2 + 34X_3 &\le 80 \\ 22X_1 + 13X_2 + 12X_3 &\le 40 \\ X_1, X_2, X_3 &= 0 \text{ or } 1 \end{aligned}$$

Now reformulate this problem with additional constraints so that no more than two of the three variables can take on a value equal to 1 in the solution. Further, make sure that if $X_1 = 1$, then $X_2 = 1$ also, and vice versa. Then solve the new problem using Excel.

Solution

We need two new constraints to handle the reformulated problem:

$$X_1 + X_2 + X_3 \le 2$$

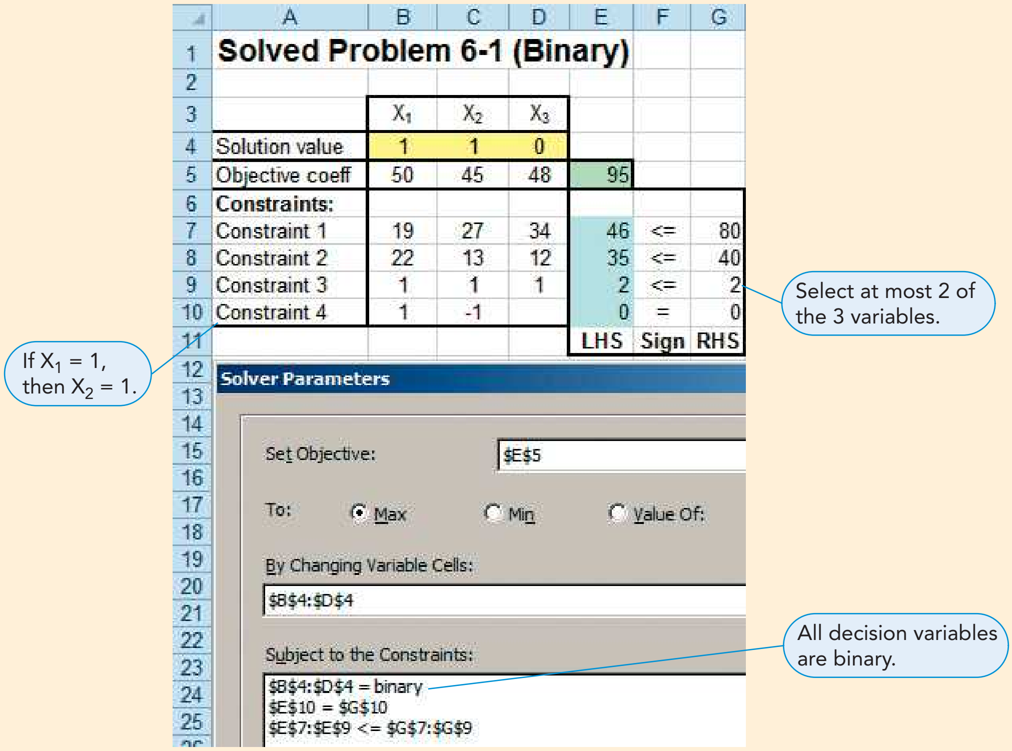
and

$$X_1 - X_2 = 0$$

File: 6-9.xls

The Excel layout and Solver entries for this problem are shown in Screenshot 6-9. The optimal solution is $X_1 = 1, X_2 = 1, X_3 = 0$, with an objective value of 95.

SCREENSHOT 6-9 **Excel Layout and Solver Entries for Solved Problem 6-1**



Solved Problem 6-2

Recall the Harrison Electric Company general IP problem discussed in section 6.2. Its IP model is

$$\text{Maximize profit} = \$600L + \$700F$$

subject to the constraints

$$\begin{aligned} 2L + 3F &\le 12 && \text{(wiring hours)} \\ 6L + 5F &\le 30 && \text{(assembly hours)} \\ L, F &\ge 0, \text{ and integer} \end{aligned}$$

where L = number of lamps produced and F = number of ceiling fans produced. Reformulate and solve Harrison Electric's problem as a GP model, with the following goals in rank order. (Note that more than one goal has been assigned the same rank.) Remember that both L and F need to be integer valued.

Rank R_1: Produce at least 4 lamps (goal 1) and 3 ceiling fans (goal 2).

Rank R_2: Limit overtime in the assembly department to 10 hours (goal 3) and in the wiring department to 6 hours (goal 4).

Rank R_3: Maximize profit (goal 5).

Solution

Let us define d_i^- and d_i^+ as the underachievement and overachievement deviation variables, respectively, for the ith goal. Then, the GP model can be formulated as follows:

$$\text{Minimize} = R_1(d_1^- + d_2^-) + R_2(d_3^+ + d_4^+) + R_3(d_5^-)$$

subject to the constraints

$$\left.\begin{aligned} L + d_1^- - d_1^+ &= 4 \\ F + d_2^- - d_2^+ &= 3 \end{aligned}\right\} \quad \text{Rank 1}$$

$$\left.\begin{aligned} 2L + 3F + d_3^- - d_3^+ &= 18 \\ 6L + 5F + d_4^- - d_4^+ &= 40 \end{aligned}\right\} \quad \text{Rank 2}$$

$$600L + 700F + d_5^- - d_5^+ = 99{,}999\} \quad \text{Rank 3}$$

$$L, F, \text{all } d_i \geq 0$$

$$L, F \text{ integer}$$

The time availabilities in the wiring and assembly departments (rank R_2 goals) have been adjusted to include the permissible overtime. The target of \$99,999 for the rank R_3 goal represents an unrealistically high profit. It is just a mathematic trick to use as a target so that we can get as close as possible to the maximum profit.

Note that for the rank R_1 goals, we need to minimize only the underachievement deviation variables d_1^- and d_2^-. Likewise, for the rank R_2 goals, we must minimize only the overachievement deviation variables d_3^- and d_4^+. Finally, for the rank R_3 goal, we must minimize only the underachievement deviation variable d_5^-.

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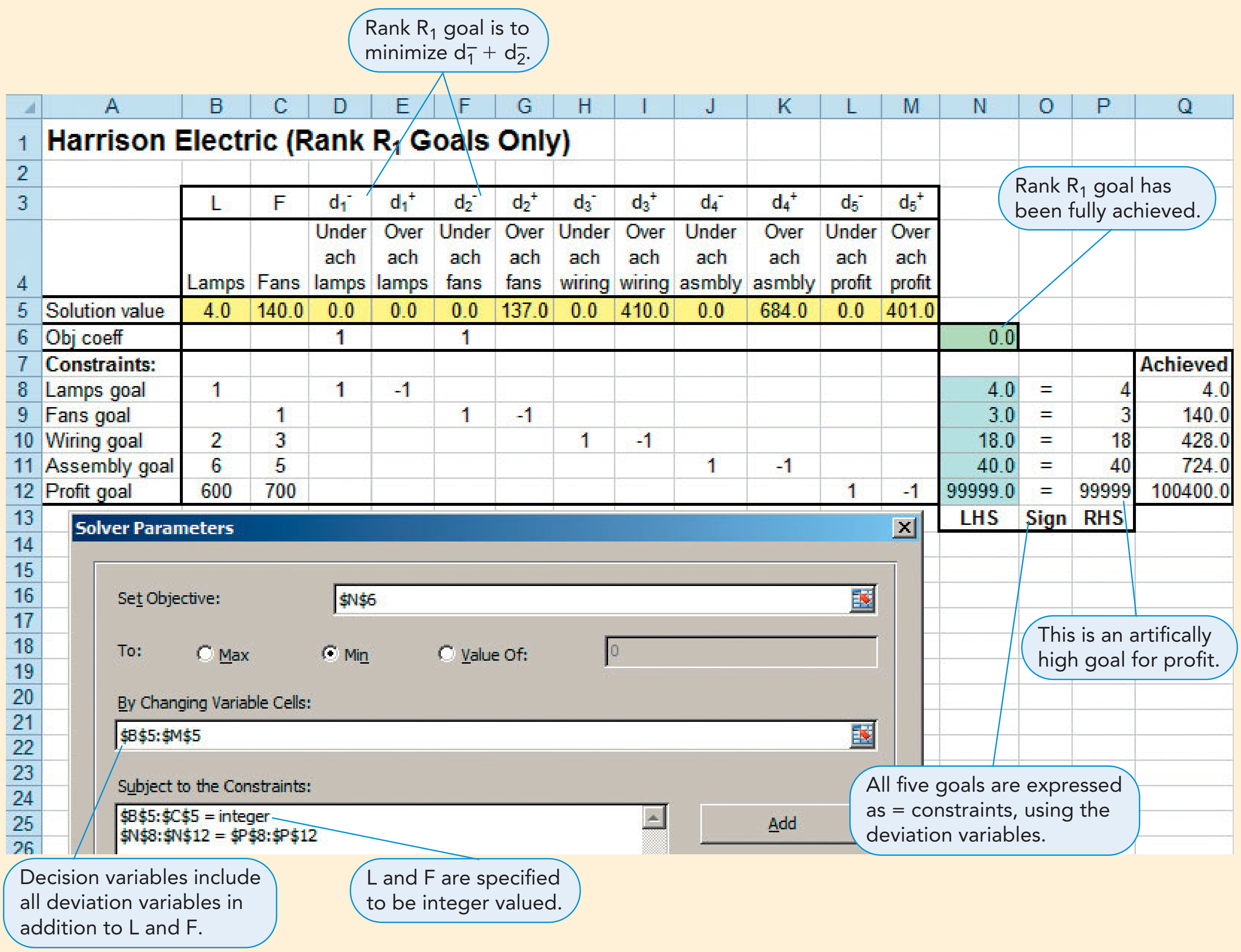
File: 6-10.xls

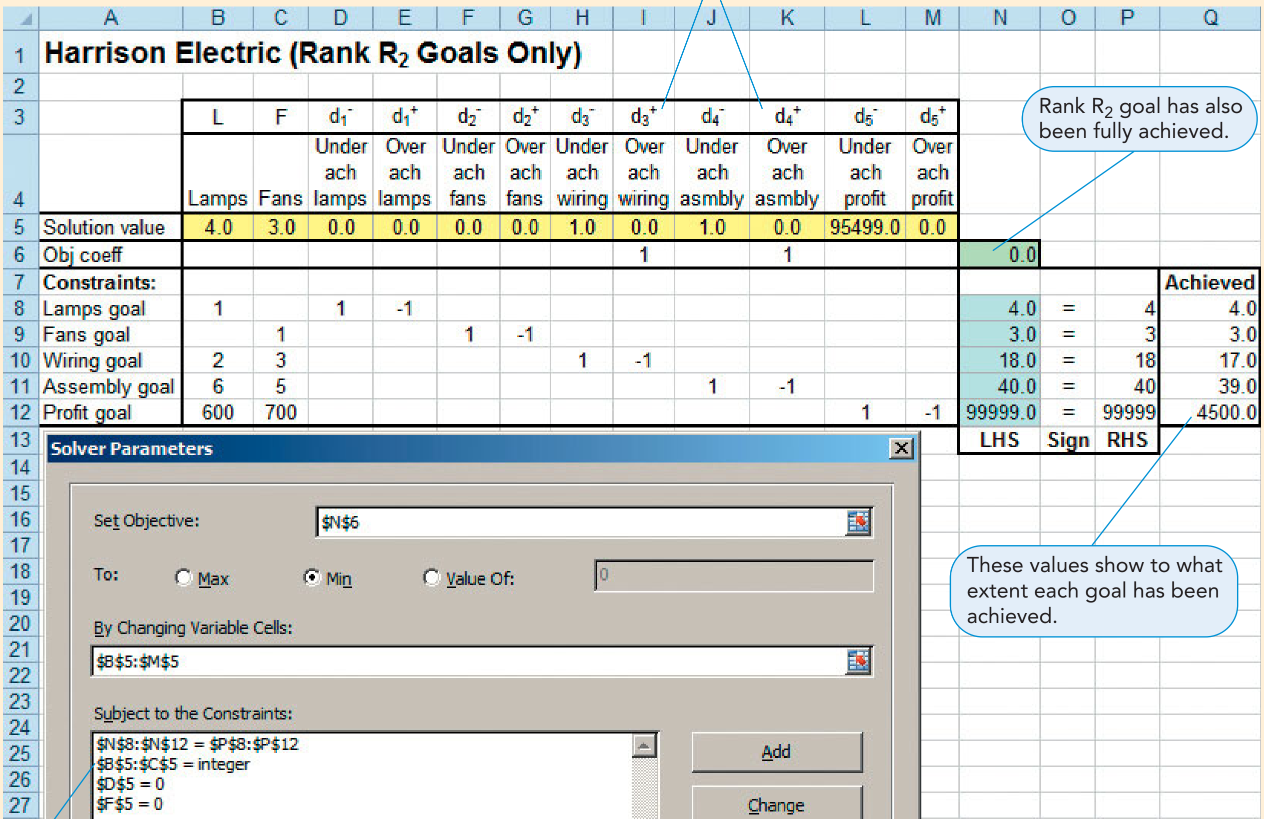
Screenshots 6-10A, 6-10B, and 6-10C show the Excel layout and Solver entries for this problem when each goal is considered in order of its rank. In each case, optimal values of the deviation variables for a higher-ranked goal are explicitly specified while solving the problem for a lower-ranked goal. For example, while solving the GP problem with the rank R_2 goals (Screenshot 6-10B), the optimal values of the rank R_1 deviation variables (namely $d_1^- = 0$ and $d_2^- + 0$) from Screenshot 6-10A are explicitly specified in Solver.

SCREENSHOT 6-10A **Excel Layout and Solver Entries for Harrison Electric—Rank R_1 Goals Only**



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	Harrison Electric (Rank R_1 Goals Only)																
2																	
3		L	F	d_1^-	d_1^+	d_2^-	d_2^+	d_3^-	d_3^+	d_4^-	d_4^+	d_5^-	d_5^+				
4		Lamps	Fans	Under ach lamps	Over ach lamps	Under ach fans	Over ach fans	Under ach wiring	Over ach wiring	Under ach asmbly	Over ach asmbly	Under ach profit	Over ach profit				
5	Solution value	4.0	140.0	0.0	0.0	0.0	137.0	0.0	410.0	0.0	684.0	0.0	401.0				
6	Obj coeff			1		1								0.0			
7	Constraints:																Achieved
8	Lamps goal	1		1	-1									4.0	=	4	4.0
9	Fans goal		1			1	-1							3.0	=	3	140.0
10	Wiring goal	2	3					1	-1					18.0	=	18	428.0
11	Assembly goal	6	5							1	-1			40.0	=	40	724.0
12	Profit goal	600	700									1	-1	99999.0	=	99999	100400.0
13														LHS	Sign	RHS	

SCREENSHOT 6-10B **Excel Layout and Solver Entries for Harrison Electric—Rank R_2 Goals Only**



Harrison Electric (Rank R_2 Goals Only)

	L	F	d_1^-	d_1^+	d_2^-	d_2^+	d_3^-	d_3^+	d_4^-	d_4^+	d_5^-	d_5^+				
	Lamps	Fans	Under ach lamps	Over ach lamps	Under ach fans	Over ach fans	Under ach wiring	Over ach wiring	Under ach asmbly	Over ach asmbly	Under ach profit	Over ach profit				
Solution value	4.0	3.0	0.0	0.0	0.0	0.0	1.0	0.0	1.0	0.0	95499.0	0.0				
Obj coeff								1		1			0.0			
Constraints:																Achieved
Lamps goal	1		1	-1									4.0	=	4	4.0
Fans goal		1			1	-1							3.0	=	3	3.0
Wiring goal	2	3					1	-1					18.0	=	18	17.0
Assembly goal	6	5							1	-1			40.0	=	40	39.0
Profit goal	600	700									1	-1	99999.0	=	99999	4500.0
													LHS	Sign	RHS	

Solver Parameters

Set Objective: \$N\$6

To: Max Min Value Of: 0

By Changing Variable Cells: \$B\$5:\$M\$5

Subject to the Constraints:

\$N\$8:\$N\$12 = \$P\$8:\$P\$12
\$B\$5:\$C\$5 = integer
\$D\$5 = 0
\$F\$5 = 0

Add
Change

The optimal solution shown in Screenshot 6-10A considers only the rank R_1 goals. Therefore, restricting overtime in assembly and wiring is not an issue in this problem. The solution therefore uses a large amount of overtime (note the values for d_3^+ and d_4^+ in Screenshot 6-10A).

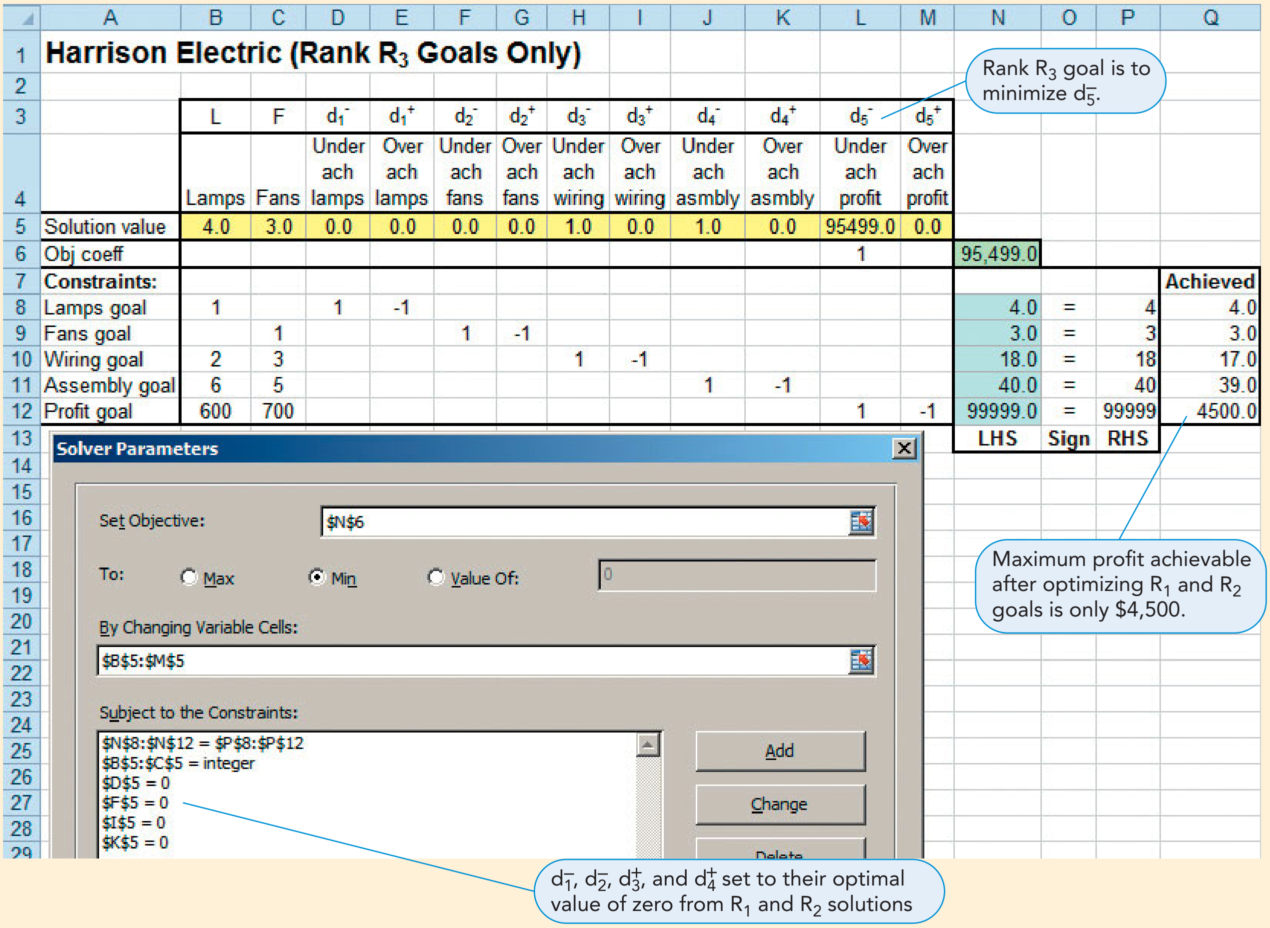
However, when we now solve the rank R_2 GP problem in Screenshot 6-10B, the solution minimizes the use of excessive overtime. As a consequence, the deviation variable for underachieving profit (d_5^-) now has a large value (due to the artificially large value of \$99,999 we used as the target for profit).

When we now try to minimize this deviation variable in the rank R_3 GP problem (Screenshot 6-10C), we obtain the overall optimal solution for this GP problem. The optimal solution is $L = 4$, $F = 3$, $d_1^- = 0, d_2^- = 0$, $d_3^+ = 0$, and $d_4^+ = 0$, and $d_5^- = 95{,}499$. In effect, this means that the maximum profit we can get while achieving our higher-ranked goals is only \$4,500 (= \$600 × 4 + \$700 × 3).

Solved Problem 6-3

Thermolock Corporation produces massive rubber washers and gaskets like the type used to seal joints on the NASA space shuttles. To do so, it combines two ingredients, rubber and oil. The cost of the industrial-quality rubber used is \$5 per pound, and the cost of the high-viscosity oil is \$7 per pound. Two of the three constraints Thermolock faces are nonlinear. If R and O denote the number of pounds used of rubber and oil, respectively, the firm's objective function and constraints can be written as follows:

SCREENSHOT 6-10C Excel Layout and Solver Entries for Harrison Electric—Rank R_3 Goals Only



$$\text{Minimize cost} = \$5R + \$7O$$

subject to the constraints

$$\begin{aligned}
3R + 0.25R^2 + 4O + 0.3O^2 &\geq 125 && \text{(hardness constraint)}\\
13R + R^3 &\geq 80 && \text{(tensile strength constraint)}\\
0.7R + O &\geq 17 && \text{(elasticity constraint)}\\
R,O &\geq 0
\end{aligned}$$

Set up and solve Thermolock's NLP model using Solver.

Solution

The Excel layout and Solver entries for Thermolock's problem are shown in Screenshot 6-11A. As in the Pickens Memorial NLP example, the only decision variables (i.e., changing variable cells) are in cells B5 and C5. The entries in row 8 (cells B8:F8) represent all linear and nonlinear terms involving these decision variables. The formulas for these cells are

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File: 6-11.xls, sheet: 6-11A

$$\begin{aligned}
&= B5 && \text{(entry for } R \text{ in cell B8)}\\
&= C5 && \text{(entry for } O \text{ in cell C8)}\\
&= B5\text{\^{}}2 && \text{(entry for } R^2 \text{ in cell D8)}\\
&= C5\text{\^{}}2 && \text{(entry for } O^2 \text{ in cell E8)}\\
&= B5\text{\^{}}2 && \text{(entry for } R \text{ in cell F8)}
\end{aligned}$$

Let us use the Multistart option available in the GRG Nonlinear method in Solver to solve this NLP model. Recall that for NLP models, we should select GRG Nonlinear as the solving method to use.

We then click the Options button in Solver and check the boxes labeled Use Multistart and Require Bounds on Variables in the GRG Nonlinear tab (refer to Screenshot 6-8D). As noted previously, the nonnegative constraints provide lower bounds for the variables. In Thermolock's case, we have specified a value of 50 as an upper bound for each of the two decision variables. When we now solve the model, we get the result shown in Screenshot 6-11A, which specifies that Thermolock should use 3.325 pounds of rubber and 14.672 pounds of oil, at a total cost of $119.33.

Here again, we ask whether this is a local or global optimal solution. Since we have used Solver to automatically try different starting values for the decision variables (within the bounds we have specified), it is probable, although not guaranteed, that this is a global optimal solution. As shown in Screenshot 6-11B, Solver makes this fact clear in its message in the Solver Results window. By the way, to verify that other local optimal solutions are possible, we urge you to use the Excel file *6-11.xls* on the Companion Website for this textbook and experiment with different starting values for *R* and *O*. For example, when we start with values of 5 each for *R* and *O*, the final result (not shown here) obtained by Solver is to use 10 pounds each of rubber and oil, at a total cost of $120. Because this cost is higher than the $119.33 cost in the earlier solution (Screenshot 6-11A), it is clear that this solution is a local optimal solution.

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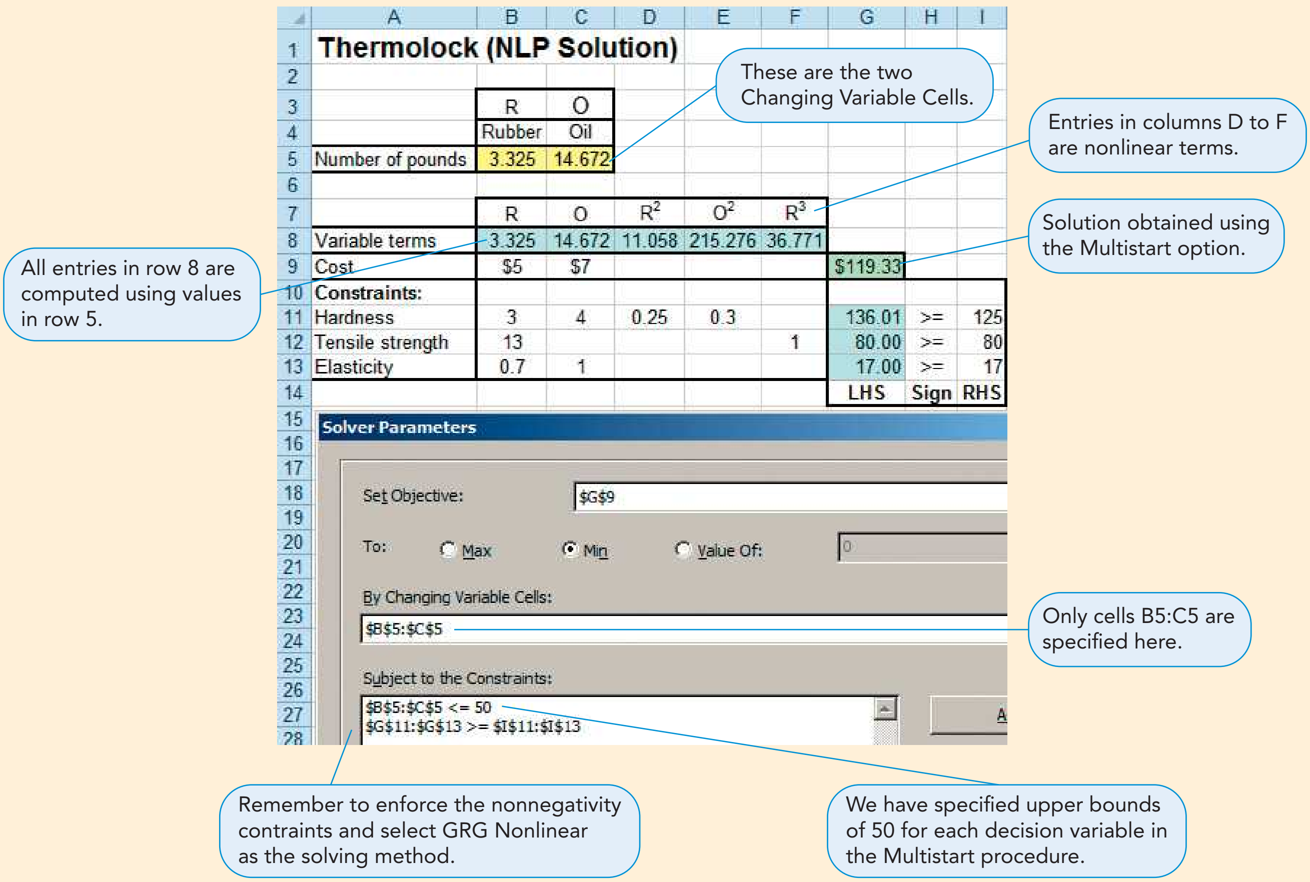
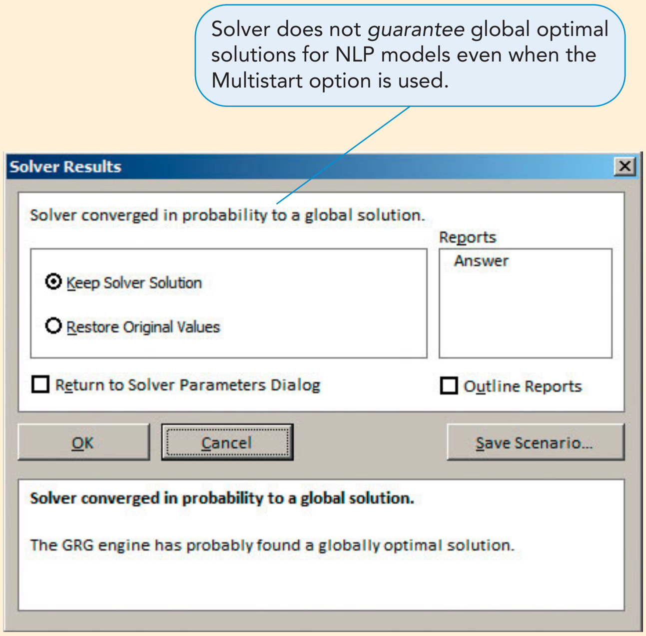
File: 6-11.xls, sheet: 6-11B

SCREENSHOT 6-11A Excel Layout and Solver Entries for Thermolock



SCREENSHOT 6-11B Solver Results Window When Using the Multistart Option for NLP Models



Discussion Questions and Problems

Discussion Questions

6-1 Compare the similarities and differences of LP and GP.

6-2 Provide your own examples of five applications of IP.

6-3 What is the difference between pure and mixed IP models? Which do you think is most common, and why?

6-4 What is meant by *satisficing*, and why is the term often used in conjunction with GP?

6-5 What are deviation variables? How do they differ from decision variables in traditional LP problems?

6-6 If you were the president of the college you are attending and were employing GP to assist in decision making, what might your goals be? What kinds of constraints would you include in your model?

6-7 What does it mean to rank goals in GP? How does this affect the problem's solution?

6-8 Provide your own examples of problems where (a) the objective is nonlinear and (b) one or more constraints are nonlinear.

6-9 Explain in your own words why IP problems are more difficult to solve than LP problems.

6-10 Explain the difference between assigning weights to goals and ranking goals.

6-11 What does the term *quadratic programming* mean?

6-12 Which of the following are NLP models, and why? Are any of these quadratic programming models?

(a) Maximize profit $= 3X_1 + 5X_2 + 99X_3$
subject to the contraints

$$X_1 \geq 10$$
$$X_2 \leq 5$$
$$X_3 \geq 18$$

(b) Maximize profit $= 25X_1 + 30X_2 + 8X_1X_2$
subject to the contraints

$$X_1 \geq 8$$
$$X_1 + X_2 \geq 12$$
$$X_1 - X_2 = 11$$

(c) Maximize profit $= 3X_1 + 4X_2$
subject to the contraints

$$X_1^2 - 5X_2 \geq 8$$
$$3X_1 + 4X_2 \geq 12$$

(d) Maximize profit $= 18X_1 + 5X_2 + X_2^2$
subject to the contraints

$$4X_1 - 3X_2 \geq 8$$
$$X_1 + X_2 \geq 18$$

Problems

6-13 A cleaning crew currently spends 6 hours per house cleaning eight houses every day, for a profit of $15 per hour. The crew now wants to offer its services to other houses, as well as small professional offices. The crew believes it will take 7 hours to clean the office of a lawyer and that the profit for doing so will be $19 per hour. It will take 10 hours to clean a doctor's office and that the profit for doing so will be $25 per hour. The crew can make 120 hours of labor available per day and does not want to cancel any of its existing house contracts. What is the best mix of homes and offices to clean per day to maximize profit? Remember that your solution must be in whole numbers.

6-14 An airline is preparing to replace its old planes with three new styles of jets. The airline needs 17 new planes to service its current routes. The decision regarding which planes to purchase should balance cost with capability factors, including the following: (1) The airline can finance up to $700 million in purchases; (2) each 7A7 jet will cost $38 million, each 7B7 jet will cost $27 million, and each 7C7 jet will cost $22 million; (3) at least one-third of the planes purchased should be the longer-range 7A7; (4) the annual maintenance budget is to be no more than $12 million; (5) the annual maintenance cost per 7A7 is estimated to be $800,000, $600,000 for each 7B7, and $500,000 for each 7C7; and (6) annually, each 7A7 can carry 125,000 passengers, each 7B7 can fly 95,000 passengers, and each 7C7 can fly 80,000 passengers. Formulate this as an IP problem to maximize the annual passenger-carrying capability. Solve it by using Excel.

6-15 The Gaubert Marketing Company needs the following number of telemarketers on the phones during the upcoming week: Monday 23, Tuesday 16, Wednesday 21, Thursday 17, Friday 20, Saturday 12, and Sunday 15. Each employee works five consecutive days followed by 2 days off per week. How many telemarketers should be scheduled each day of the week to begin their five-day work week? The objective is to minimize the total number of employees needed to fulfill the daily requirements.
(a) Solve as an IP model.
(b) Additional information is now available for Gaubert. Daily pay from Monday through Friday is $90, pay for Saturday is $110, and Sunday workers earn $125. In addition, up to four people can be hired who will work Friday, Saturday, and Sunday. Their pay for this three-day week is $250. The new objective is to minimize total weekly labor costs. Revise the IP model and solve it.

6-16 A hospital is planning an $8 million addition to its existing facility. The architect has been asked to consider the following design parameters: (1) There should be at least 10 and no more than 20 intensive care unit (ICU) rooms; (2) there should be at least 10 and no more than 20 cardiac care unit (CCU) rooms; (3) there should be no more than 50 double rooms; (4) there should be at least 35 single rooms; and (5) all patient rooms should fit inside the allotted 40,000-square-foot space (not including hallways). The following table summarizes the relevant room data:

	SINGLE	DOUBLE	ICU	CCU
Cost per room to build and furnish ($thousands)	$45	$54	$110	$104
Minimum square feet required	300	360	320	340
Profit per room per month ($thousands)	$21	$28	$ 48	$ 41

How many rooms of each type should the architect include in the new hospital design?

6-17 A vending machine is programmed to count out the correct change for each transaction. Formulate and solve an IP model that will determine how change is to be made for a purchase of $4.43, when a $10 bill is inserted into the machine. The model's solution should be based on the availability of coins in the machine, with the objective of minimizing the total number of coins used to make the change.

DENOMINATION	AVAILABILITY
$1 coin	8
Quarter ($0.25)	9
Dime ($0.10)	7
Nickel ($0.05)	11
Penny ($0.01)	10

6-18 Stockbroker Susan Drexler has advised her client as shown in the following table.

INVESTMENT	COST (THOUSANDS)	EXPECTED RETURN (THOUSANDS)
Andover municipal bonds	$ 400	$ 35
Hamilton city bonds	$1,000	$100
East Power & Light Co.	$ 350	$ 30
Nebraska Electric Service	$ 700	$ 65
Southern Gas and Electric	$ 490	$ 45
Manuel Products Co.	$ 270	$ 20
Builders Paint Co.	$ 800	$ 90
Rest Easy Hotels Co.	$ 500	$ 50

The client agrees to this list but provides several conditions: (1) No more than $3,000,000 can be invested, (2) the money is to be spread among at least five investments, (3) no more than one type of bond can be purchased, (4) at least two utility stocks must be purchased, and (5) at least two regular stocks must be purchased. Formulate this as a 0–1 IP problem for Ms. Drexler to maximize expected return. Solve it by using Excel.

6-19 Porter Investments needs to develop an investment portfolio for Mrs. Singh from the following list of possible investments:

INVESTMENT	COST	EXPECTED RETURN
A	$ 10,000	$ 700
B	$ 12,000	$ 1,000
C	$ 3,500	$ 390
D	$ 5,000	$ 500
E	$ 8,500	$ 750
F	$ 8,000	$ 640
G	$ 4,000	$ 300

Mrs. Singh has a total of $60,000 to invest. The following conditions must be met: (1) If investment F is chosen, then investment G must also be part of the portfolio, (2) at least four investments should be chosen, and (3) of investment A and B, exactly one must be included. What stocks should be included in Mrs. Singh's portfolio?

6-20 A truck with the capacity to load 2,200 cubic feet of cargo is available to transport items selected from the following table.

ITEM	VALUE	VOLUME (CU. FT.)
A	$1,800	700
B	$1,400	600
C	$1,100	450
D	$ 900	400
E	$1,600	650
F	$1,100	350
G	$1,200	600

If selected, an item must be shipped in its entirety (i.e., partial shipments are not allowed). Of items B, C, and D, at least two items must be selected. If item B is selected, then item G cannot be selected. Which items should be selected to maximize the value of the shipment?

6-21 The Greenville Ride have $19 million available to sign free agent pitchers for the next season. The following table provides the relevant information for eight pitchers who are available for signing, such as whether each throws right or left handed, whether each is a starter or reliever, the cost in millions of dollars to sign each, and the relative value of each on the market on a scale of 1 to 10 (10 = highest).

PITCHER	THROWS	START/ RELIEF	COST (MILLIONS)	VALUE
A	L	R	$9	8
B	R	S	$4	5
C	R	S	$5	6
D	L	S	$5	5
E	R	R	$6	8
F	R	R	$3	5
G	L	S	$8	7
H	R	S	$2	4

The Ride feel the following needs exist for next season: (1) at least two right-handed pitchers, (2) at least one left-handed pitcher, (3) at least two starters, and (4) at least one right-handed reliever. Who should the Ride try to sign, if their objective is to maximize total value?

6-22 Allied Products has six R&D projects that are potential candidates for selection during the upcoming fiscal year. The table at the bottom of the page provides the expected net present value (NPV) and capital requirements over the next five years for each project.

The table also indicates the planned budget expenditures for the entire R&D program during each of the next five years. Which projects should be selected?

Table for Problem 6-22

		CAPITAL REQUIRED (THOUSANDS)				
PROJECT	NPV (THOUSANDS)	YEAR 1	YEAR 2	YEAR 3	YEAR 4	YEAR 5
1	$140	$ 80	$25	$22	$18	$10
2	$260	$ 95	$40	$ 5	$10	$35
3	$ 88	$ 58	$17	$14	$12	$12
4	$124	$ 32	$24	$10	$ 6	$ 7
5	$176	$115	$25	$25	$10	$ 0
6	$192	$ 48	$20	$12	$32	$40
R&D Budget		$225	$80	$60	$50	$50

Table for Problem 6-23

	ATLANTA	BOSTON	CHICAGO	DALLAS	DENVER	LA	PHILADELPHIA	SEATTLE
Atlanta	—	1,108	717	783	1,406	2,366	778	2,699
Boston		—	996	1,794	1,990	3,017	333	3,105
Chicago			—	937	1,023	2,047	767	2,108
Dallas				—	794	1,450	1,459	2,112
Denver					—	1,026	1,759	1,313
Los Angeles						—	2,723	1,141
Philadelphia							—	2,872

6-23 I-Go Airlines has operations in eight cities throughout the United States. It is searching for the best location(s) to designate as a hub, which would then serve other cities within a 1,400-mile radius. For economic reasons, I-Go would like to operate no more hubs than necessary to cover all eight cities. Which of the cities in the table at the top of the page should be designated as hubs?

6-24 Laurens County has six communities that need to be served by fire stations. The number of minutes it takes to travel between the communities is shown in the following table. The county would like to establish the minimum number of fire stations so that each community can get a response in five minutes or less. How many stations will be needed, and what communities will each station serve?

	A	B	C	D	E	F
A	—	4	6	3	5	8
B		—	4	10	6	5
C			—	9	3	5
D				—	6	3
E					—	10

6-25 Georgia Atlantic Corporation needs to decide on the locations for two new warehouses. The candidate sites are Philadelphia, Tampa, Denver, and Chicago. The following table provides the monthly capacities and the monthly fixed costs for operating warehouses at each potential site.

WAREHOUSE	MONTHLY CAPACITY (UNITS)	MONTHLY FIXED COST
Philadelphia	250	$1,000
Tampa	260	$ 800
Chicago	280	$1,200
Denver	270	$ 700

The warehouses will need to ship to three marketing areas: North, South, and West. Monthly requirements are 200 units for North, 180 units for South, and 120 units for West. The following table provides the cost to ship one unit between each location and destination.

WAREHOUSE	NORTH	SOUTH	WEST
Philadelphia	$4	$ 7	$ 9
Tampa	$6	$ 3	$11
Chicago	$5	$ 6	$ 5
Denver	$8	$10	$ 2

In addition, the following conditions must be met by the final decision: (1) A warehouse must be opened in either Philadelphia or Denver, and (2) if a warehouse is opened in Tampa, then one must also be opened in Chicago. Which two sites should be selected for the new warehouses to minimize total fixed and shipping costs?

6-26 A manufacturer has acquired four small assembly plants, located in Charlotte, Tulsa, Memphis, and Buffalo. The plan is to remodel and keep two of the plants and close the other two. The table at the top of the next page provides the anticipated monthly capacities and the monthly fixed costs for operating plants at each potential site. It is estimated that the costs to remodel and/or close the plants are equivalent.

Because of union considerations, if the plant in Buffalo is kept open, the plant in Tulsa must also be kept open. If the objective is to minimize total fixed and shipping costs, which two sites should be selected to continue assembly?

6-27 A hospital in a large city plans to build two satellite trauma centers to provide improved emergency service to areas, such as highways and high-crime districts, that have historically demonstrated an increased need for critical care services. The city council has identified four potential locations for these centers. The table for Problem 6-27 near the top of the next page shows the locations (1–4), and the mileage from each center to each of the eight high-need areas (A–H). Assume that the cost of building the trauma centers will be the same, regardless of which locations are chosen.